EXERCISES MATH 202B - Fourth Assignment

1. Let D_n be the dihedral group given by generators a and b and relations $a^n = 1 = b^2$ and $bab = a^{-1}$. Moreover, define maps $r_a, r_b : \mathbf{C} \to \mathbf{C}$ by $r_a z = \theta z$ and $r_b z = \bar{z}$, where $\theta \in \mathbf{C}$. Moreover, we say that a map $f : \mathbf{C} \to \mathbf{C}$ is $\mathbf{R}$-linear if $f(t_1 z_1 + t_2 z_2) = t_1 f(z_1) + t_2 f(z_2)$ for all $t_1, t_2 \in \mathbf{R}$ and $z_1, z_2 \in \mathbf{C}$. Show:
(a) Both r_a and r_b are $\mathbf{R}$-linear.
(b) Determine all possible $\theta \in \mathbf{C}$ for which $a \mapsto r_a$, $b \mapsto r_b$ is a representation of D_n.
(c) Write down the matrices of r_a and r_b for the representations in (b) with respect to the $\mathbf{R}$-basis $\{1, i\}$ (the matrices need to have real entries).
(d) Calculate the characters with respect to each of these representations.
(e) Which of these representations are isomorphic?
(f) Which of these representations are simple?
(g) Give a complete list of all simple D_n-modules, up to isomorphism. Prove that these are all.

2. Let $V = \mathbf{C}^n$ be the n-dimensional S_n-module, where the action is given by permuting the standard basis vectors and let χ_V be its character.
(a) Let $\varepsilon(\pi) = \det(\rho_V(\pi))$. Show: If χ is a character of S_n, then so is $\varepsilon\chi$, defined by $(\varepsilon\chi)(\pi) = \varepsilon(\pi)\chi(\pi)$, and χ is simple if and only if $\varepsilon\chi$ is simple.
(b) From now on let $n = 4$. Determine all the conjugacy classes of S_4, and how many elements each of them contains.
(c) Compute $\langle \chi_V, \chi_V \rangle$. Write χ_V as a sum of simple characters, and compute their values at all the conjugacy classes.
(d) Write $\varepsilon\chi_V$ as a sum of simple characters, and compute their values at all conjugacy classes.
(e) Deduce the character table of S_4 from (a)-(e). (Suggestion: After determining how many more simple characters there are besides the ones in (c) and (d), use the formula $\chi_{reg} = \sum_\lambda d_\lambda \chi_\lambda$, where d_λ is the dimension of the simple module V_λ.)

You may use the following facts which will be defined and proved in the course: If χ, ψ are characters of G, one defines

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \bar{\psi}(g).$$

Then we have that χ is the character of a simple representation if and only if $\langle \chi, \chi \rangle = 1$. More generally, let W be a representation with multiplicities m_λ (i.e. if W is written as a direct sum of simple modules, m_λ of them are $\cong V_\lambda$). Then $\langle \chi, \chi \rangle = \sum_\lambda m_\lambda^2$ and $m_\lambda = \langle \chi, \chi_\lambda \rangle$, where χ_λ is the character of the simple G-module V_λ.