EXERCISES MATH 202B - Fifth Assignment

1. Let $H$ be a normal subgroup of the group $G$ (i.e. $gHg^{-1} = H$ for all $g \in G$), and let $V$ be a simple $G$-module. Then $V$, considered as an $H$-module, may or may not be simple. Show that if $W$ and $W'$ are simple $H$-submodules of $V$, then they must have the same dimensions (Hint: Show that one can write $V$ as a direct sum of simple $H$-modules of the form $gW$ for certain $g \in G$).

2. Let $A_4 = \{ \pi \in S_4, \varepsilon(\pi) = 1 \}$, with $\varepsilon$ as in Problem 2 of the fourth assignment. As $\varepsilon$ is a group homomorphism, $A_4$ is a normal subgroup of $S_4$.
   (a) Show that $A_4$ has an abelian normal subgroup $V$ of order four.
   (b) What are the possible dimensions for simple $A_4$-modules? How does $CA_4$ decompose into simple matrix rings?
   (c) Determine all simple characters of $A_4$ via induction from representations of $V$.

3. Let $V$ be a $G$-module with character $\chi$. Assume that $\chi(g) \in \mathbb{R}$ for all $g \in G$. Show:
   (a) The trivial representation appears in $V^\otimes 2$ exactly $\sum \lambda m_\lambda^2$ times, if $V = \bigoplus \lambda V_\lambda^{m_\lambda}$.
   (b) If $V, W$ are simple $G$-modules, the trivial representation appears in $V \otimes W$ if and only if $V \cong W$.
   (c) Find a counter example to (a) if $\chi(g) \not\in \mathbb{R}$ for some $g \in G$.

4. Let $V$ be the 3-dimensional irreducible $S_4$-submodule of its permutation representation.
   (a) Find the decomposition of $V \otimes V$ into a direct sum of simple $S_4$-modules.
   (Hint: Knowing the dimensions of simple $S_4$-modules, it should suffice to only work with the character of $V$. But you can use the explicit $S_4$ characters from the previous homework.)
   (b) Let $W$ be the two-dimensional simple representation of $S_4$. How does $V \otimes W$ decompose? (Hint: It should be enough to calculate $\langle \chi_V \chi_W, \chi_V \chi_W \rangle$.)

Remark You may want to check for yourself that it is fairly easy now to determine the decomposition of the tensor product of any two simple representations of $S_4$ from the computations in (a) and (b). There are still no simple rules known how to determine the decomposition of two arbitrary simple representations of $S_n$ for $n$ large.