EXERCISES MATH 202B - Fifth Assignment

1. Let H be a normal subgroup of the group G (i.e. $gHg^{-1} = H$ for all $g \in G$), and let V be a simple G-module. Then V, considered as an H-module, may or may not be simple. Show that if W and W' are simple H-submodules of V, then they must have the same dimensions (*Hint*: Show that one can write V as a direct sum of simple H-modules of the form $g.W$ for certain $g \in G$).

2. Let $A_4 = \{\pi \in S_4,\ \varepsilon(\pi) = 1\}$, with ε as in Problem 2 of the fourth assignment. As ε is a group homomorphism, A_4 is a normal subgroup of S_4.
(a) Show that A_4 has an abelian normal subgroup V of order four.
(b) What are the possible dimensions for simple A_4-modules? How does $\mathbf{C}A_4$ decompose into simple matrix rings?
(c) Determine all simple characters of A_4 via induction from representations of V.

3. Let V be a G-module with character χ. Assume that $\chi(g) \in \mathbf{R}$ for all $g \in G$. Show:
(a) The trivial representation appears in $V^{\otimes 2}$ exactly $\sum_\lambda m_\lambda^2$ times, if $V = \oplus_\lambda V_\lambda^{m_\lambda}$.
(b) If V, W are simple G-modules, the trivial representation appears in $V \otimes W$ if and only if $V \cong W$.
(c) Find a counter example to (a) if $\chi(g) \notin \mathbf{R}$ for some $g \in G$.

4. Let V be the 3-dimensional irreducible S_4-submodule of its permutation representation.
(a) Find the decompostion of $V \otimes V$ into a direct sum of simple S_4-modules.
(*Hint*: Knowing the dimensions of simple S_4-modules, it should suffice to only work with the character of V. But you can use the explicit S_4 characters from the previous homework.)
(b) Let W be the two-dimensional simple representation of S_4. How does $V \otimes W$ decompose? (*Hint*: It should be enough to calculate $\langle \chi_V\chi_W, \chi_V\chi_W\rangle$.)

   Remark You may want to check for yourself that it is fairly easy now to determine the decomposition of the tensor product of any two simple representations of S_4 from the computations in (a) and (b). There are still no simple rules known how to determine the decomposition of two arbitrary simple representations of S_n for n large.