EXERCISES MATH 202B - Seventh Assignment

1. Let A be a semisimple algebra over $\mathbf{C}$ with $A = \oplus_\lambda A_\lambda$ the decomposition into a direct sum of simple algebras. Let $V = \bigoplus_\lambda V_\lambda^{m_\lambda}$ be the decomposition of V into a direct sum of simple A-modules. Moreover, let e_λ be a minimal idempotent in A_λ. Show that $m_\lambda = Tr_V(\rho(e_\lambda))$, where $\rho(e_\lambda)$ is the linear map via which e_λ acts on V, and Tr_V is the usual trace on End(V). (In future, we may just write $Tr_V(e_\lambda)$ instead of $Tr_V(\rho(e_\lambda))$.

2. Fix a Young tableau t of shape μ. Let $M^\mu = \mathbf{C}S_n p_t$ be the representation induced from the trivial representation of the row stablizer P_t of t, and let $S^\mu = \mathbf{C}S_n q_t p_t$. Moreover, let $M^\mu = \oplus_\lambda (S^\lambda)^{K_{\lambda\mu}}$ be a decomposition of M^μ into a direct sum of simple S_n-modules.
(a) Show that $K_{\mu\mu} = 1$ and $K_{\lambda\mu} \neq 0$ only if $\lambda_1 + \ ... \ + \lambda_i \geq \mu_1 + \ ... \ + \mu_i$ for $i = 1, 2, \ ... \ n$. (*Hint*: By a previous homework problem, $q_s p_s$ is an idempotent up to a multiple, for any tableau s).
(b) Calculate all $K_{\lambda\mu}$ for $\mu = [2,1,1]$ and for $\mu = [2,2]$.

3. Show that all simple characters of S_n are integer valued, for all $n \in \mathbf{N}$.

4. Let $\alpha \in \mathbf{Z}_{+o}^N$ with $|\alpha| = \sum \alpha_i = n$, where $\mathbf{Z}_{+o}$ denotes the nonnegative integers. Moreover, let V be an N-dimensional vector space with basis $\{v_1, v_2, \ ... \ v_N\}$. Define the vector
$$v^\alpha = v_1^{\otimes\alpha_1} \otimes \ ... \ \otimes v_N^{\otimes\alpha_N},$$
where $v_i^{\otimes m} = v_i \otimes \ ... \ \otimes v_i$ (m factors). Let, as usual, the symmetric group S_n act on $V^{\otimes n}$ via permuting the factors of $V^{\otimes n}$. Let V^α be the S_n-module generated by v^α.
(a) What is the decomposition of V^α into simple S_n-modules? (You may use quantities defined here without having to explicitly calculate them).
(b) If e_λ is a minimal idempotent in $(\mathbf{C}S_n)_\lambda$, and $d = \mathrm{diag}(x_1, \ ..., \ x_N)$ is the diagonal matrix as in the last homework, calculate the trace of $d_{|e_\lambda V^{\otimes n}}$ in terms of quantities defined above.
(c) Calculate the result explicitly for $\lambda = [2,1,1]$ and $N = 3$.