1. In the following we consider symmetric polynomials in \( N \) variables with all Young diagrams having at most \( N \) rows. Let \( E_j \) be the \( j \)-th elementary symmetric function, i.e.

\[
E_j = \sum_{1 \leq i_1 < \ldots < i_j \leq k} x_{i_1} x_{i_2} \ldots x_{i_j}.
\]

Define for any Young diagram \( \lambda \) the polynomial \( E_\lambda \) to be the product of the \( E_j \)'s corresponding to the columns of \( \lambda \) (e.g. \( E_{[2,3,1]} = E_3(E_2)^2 \)).

(a) Show that \( E_\lambda = x^\lambda + \text{lower terms} \).
(b) Show that the \( E_\lambda \) form a basis for the symmetric polynomials in \( N \) variables.
(c) Prove the ‘fundamental theorem of symmetric functions’: The symmetric polynomials over \( \mathbb{Z} \) are isomorphic to the polynomial ring \( \mathbb{Z}[y_1, \ldots, y_k] \) in \( k \) variables. (Hint: Show that the map \( y_k \mapsto E_i \) induces this isomorphism).

2. Let \( d \) be the \( N \times N \) diagonal matrix with diagonal entries \( x_1, x_2, \ldots, x_N \). We have calculated \( Tr_{V^\otimes n}(\pi d) \) for any permutation \( \pi \) in the lecture. You can use the following theorem which we will prove later:

**Frobenius’ Theorem:** The character \( \chi_\lambda(\pi) \) of the permutation \( \pi \) in the simple representation labeled by the Young diagram \( \lambda \) with \( \leq N \) rows is equal to the coefficient of \( x^{\lambda+\rho} \) in \( Tr_{V^\otimes n}(\pi d) \Delta \), where \( \rho_i = N - i \), and where \( \Delta = \prod_{1 \leq i < j \leq N} (x_i - x_j) \).

(a) Calculate \( \chi_{[2,2,1]}((123)(45)) \).
(b) Calculate the \( S_n \) character \( \chi_\lambda(\pi) \) for all Young diagrams \( \lambda \), where \( \pi \) is a full \( n \)-cycle. (Hint: Show first that if \( \lambda \) is not a hook diagram (hook diagrams means it only has boxes in the the first row or the first column), then \( \chi_\lambda(\pi) = 0 \).

3. Let \( \dim V = N \), with \( \{v_1, v_2, \ldots, v_N\} \) a basis for \( V \), and let \( \alpha \in \mathbb{N}^N \) and \( V^\alpha \) be as defined in the lecture. Moreover, let \([1^n]\) denote the Young diagram with all of its \( n \) boxes in one column. Let \( q = q_{[1^n]} = \sum_{\sigma \in S_n} \varepsilon(\sigma) \sigma \).

(a) Show that \( q(w_1 \otimes w_2 \otimes \ldots \otimes w_n) = 0 \) if \( w_1, w_2, \ldots, w_n \) are linearly dependent. (Hint: It is enough to show this assuming that two of the vectors are equal, by linearity.)
(b) Calculate the dimension of \( qV^\alpha \) for all possible \( \alpha \). Prove that \( Tr_{V^\otimes n}(qd) = E_n(x_1, x_2, \ldots, x_N) \), where \( d = \text{diag}(x_1, \ldots, x_N) \).
(c) Let \( t \) be a tableau of shape \( \lambda \) and let \( q_t = \sum_{\sigma \in Q_t} \varepsilon(\sigma) \sigma \), where \( Q_t \) is the column stabilizer of \( t \). Show that \( q_t V^\otimes n = 0 \) if the number of rows of \( \lambda \) is greater than \( N \).

**Remark** It is possible to reprove the combinatorial lemmas about the action of \( q_t \) on \( M^\mu \) by looking at the action of \( q_t \) on \( V^\otimes n \).