EXERCISES MATH 202B - Ninth Assignment

1. Let $s_\lambda(x_1, \ldots, x_N)$ be the Schur function corresponding to the Young diagram $\lambda$. Then also the product $s_\lambda(x_1, \ldots, x_N)(x_1 + \ldots + x_N)$ is a symmetric function and hence must be a linear combination of Schur functions. Calculate this linear combination. (Hint: Multiply by $\Delta$ and use results about antisymmetric functions).

2. (a) Let $\epsilon_\lambda$ be a minimal idempotent in $(\mathbb{C}S_n)_\lambda$ and let $\mu$ be a Young diagram with $n+1$ boxes. Calculate $\chi_\mu(\epsilon_\lambda)$, with $\epsilon_\lambda$ viewed as an element in $\mathbb{C}S_{n+1}$. (Hint: Calculate $Tr_{V^\otimes n+1}(\epsilon_\lambda d)$ for $d$ a diagonal matrix with eigenvalues $x_1, \ldots, x_N$; how is this related to $Tr_{V^\otimes n}(\epsilon_\lambda d)$?)
   
   (b) How does the simple $S_{n+1}$-module $S^\mu$ decompose as a direct sum of simple $S_n$-modules? Do NOT use the Murnaghan-Nakayama rule stated below.

   We shall prove the Murnaghan-Nakayama rule this coming week. This helps to calculate characters of $S_n$ as follows: Let $\pi$ be a permutation, and let $\pi'$ be the permutation obtained from $\pi$ by removing an $h$-cycle. Then we have
   
   $$\chi_\lambda(\pi) = \sum_\mu (-1)^{r(\mu)-1} \chi_\mu(\pi');$$

   here the summation goes over all Young diagrams $\mu$ which can be obtained from $\lambda$ by removing a rim hook of length $h$ from $\lambda$ and $r(\mu)$ is the number of rows of the rim hook.

3. Calculate the $S_{11}$ character $\chi_\lambda((1234)(567))$ for $\lambda = [6, 4, 1]$ (unlisted numbers remain fixed in the definition of the permutation).