1. Show that the Kostka number $K_{\mu \lambda}$ is equal to 1 for all shapes $\mu$. (Hint: Show that $q_\mu C_{S_n \rho_\mu}$ is equal to $C_{q_\mu \rho_\mu}$; check various lemmas we have done in class).

2. Let $\dim V = k$, with $\{v_1, v_2, \ldots, v_k\}$ a basis for $V$, and let $\alpha \in \mathbb{N}^k$ and $W_\alpha$ be as defined in the lecture. Moreover, let $[1^n]$ denote the Young diagram with all of its $n$ boxes in one column. Let $q = q_{[1^n]} = \sum_{\sigma \in S_n} \varepsilon(\sigma) \sigma$.

(a) Show that $q(w_1 \otimes w_2 \otimes \ldots \otimes w_n) = 0$ if $w_1, w_2, \ldots, w_n$ are linearly dependent. (Hint: It is enough to show this assuming that two of the vectors are equal, by linearity).

(b) Calculate the dimension of $qW_\alpha$ for all possible $\alpha$ and calculate $s_{[1^n]}(x_1, x_2, \ldots, x_k) = Tr_{V^{\otimes n}}(gg)$, where $g = \text{diag}(x_1, \ldots, x_k)$.

(c) Show that $p_\mu q_\mu V^{\otimes n} = 0$ if the number of rows of $\mu$ is bigger than the dimension of $V$.

(d) Show directly that $q_\mu W_\alpha = 0$ if the shape of $\alpha$ is larger in lexicographical order than $\mu$ (partial credit if you use theorems in the lecture).

(e) Let $g = \text{diag}(x_1, \ldots, x_k)$, and let $\tilde{s}_\mu = Tr_{V^{\otimes n}}((1/\alpha_\mu)p_\mu q_\mu g)$, where $\alpha_\mu$ is the scalar such that $(1/\alpha_\mu)p_\mu q_\mu$ is an idempotent. Show that the coefficient of $x^\alpha$ in $\tilde{s}_\mu$ is given by the Kostka number $K_{\mu \lambda(\alpha)}$.

Remark: We will show that the functions $\tilde{s}_\mu$ can be given as a quotient of two determinants, and that they are irreducible characters of the group of invertible $k \times k$ matrices.