EXERCISES MATH 202B - 8th Assignment

1. Show that the Kostka number $K_{\mu\mu}$ is equal to 1 for all shapes μ. (Hint: Show that $q_\mu \mathbf{C} S_n p_\mu$ is equal to $\mathbf{C} q_\mu p_\mu$; check various lemmas we have done in class).

2. Let dim $V = k$, with $\{v_1, v_2, \ ..., \ v_k\}$ a basis for V, and let $\alpha \in \mathbf{N}^k$ and W_α be as defined in the lecture. Moreover, let $[1^n]$ denote the Young diagram with all of its n boxes in one column. Let $q = q_{[1^n]} = \sum_{\sigma \in S_n} \varepsilon(\sigma)\sigma$.

(a) Show that $q(w_1 \otimes w_2 \otimes \ ... \ w_n) = 0$ if $w_1, w_2, \ ... \ w_n$ are linearly dependent. (Hint: It is enough to show this assuming that two of the vectors are equal, by linearity).

(b) Calculate the dimension of qW_α for all possible α and calculate $\tilde{s}_{[1^n]}(x_1, x_2, \ ..., \ x_k) = Tr_{V^{\otimes n}}(qg)$, where $g = diag(x_1, \ ... \ x_k)$.

(c) Show that $p_\mu q_\mu V^{\otimes n} = 0$ if the number of rows of μ is bigger than the dimension of V.

(d) Show directly that $q_\mu W_\alpha = 0$ if the shape of α is larger in lexicographical order than μ (partial credit if you use theorems in the lecture).

(e) Let $g = diag(x_1, \ ... \ x_k)$, and let $\tilde{s}_\mu = Tr_{V^{\otimes n}}((1/\alpha_\mu)p_\mu q_\mu g)$, where α_μ is the scalar such that $(1/\alpha_\mu)p_\mu q_\mu$ is an idempotent. Show that the coefficient of x^α in $\tilde{s}_\mu$ is given by the Kostka number $K_{\mu\lambda(\alpha)}$.

Remark We will show that the functions $\tilde{s}_\mu$ can be given as a quotient of two determinants, and that they are irreducible characters of the group of invertible $k \times k$ matrices.