EXERCISES MATH 202C - 2nd Assignment

1. Prove that

$$s_\lambda(x_1, x_2, \ ..., \ x_k) = \sum_\mu s_\mu(x_1, \ ..., \ x_{k-1}) x_k^{|\lambda|-|\mu|};$$

here $|\lambda|$ is the number of boxes of λ and the summation goes over all Young diagrams μ satisfying $\lambda_{i+1} \le \mu_i \le \lambda_i$, for $i = 1, 2, \ ..., k-1$. Here are a number of hopefully useful hints:

(a) Show that $s_\lambda(x_1, x_2, \ ..., \ x_k) = (x_1 x_2 \ ... \ x_k)^{\lambda_k} s_{\tilde\lambda}$, where $\tilde\lambda_i = \lambda_i - \lambda_k$.

(b) If $\lambda_k = 0$, show that $A_\lambda = det(x_j^{\ell_i} - x_k^{\ell_i})$, with $\ell_i = \lambda_i + k - i$ and with $1 \le i, j \le k-1$. *Remark*: This formula is useful for finding a combinatorial interpretation of Kostka numbers and for restriction rules of representations of $Gl(k)$ to $Gl(k-1)$.

2. (a) Using the restriction formula for representations of S_n, show that the dimension of the irreducible S_n-module V_λ is equal to the number of standard tableaux of shape λ.

(b) The Bratteli diagram for the algebras $\mathbf{C}S_n$, $n \in \mathbf{N}$ is given as follows: Draw all Young diagrams with n boxes in the n-th line, and connect a diagram μ on line $n-1$ with a diagram λ on line n if and only if $\mu \subset \lambda$. Show that the dimension of V_λ is equal to the number of paths from [1] to λ.

(c) Calculate the number of all paths of length $2n$ going from [1] down to the n-th line and back to [1], not necessarily the same way (each connecting line segment between two Young diagrams as in (b) has length 1).

3. Let $d_\lambda = \dim V_\lambda$, where V_λ is the irreducible representation of S_n corresponding to the Young diagram λ. What is $\sum_{\mu \supset \lambda} d_\mu$, where the summation goes over all Young diagrams μ obtained by adding a box to λ. *Hint* : For my favorite solution, consider induced representations.

4. In the following we consider symmetric polynomials in k variables with all Young diagrams having at most k rows (we set $s_\lambda = 0$ if λ has more than k rows). We have already seen that the Schur function $s_{[1^r]}$ is equal to the r-th elementary symmetric function

$$e_r = \sum_{1 \le i_1 < \ ... \ < i_r \le k} x_{i_1} x_{i_2} \ ... \ x_{i_r}.$$

Define for any Young diagram λ the polynomial e_λ to be the product of the e_r's corresponding to the columns of λ (e.g. $e_{[3,3,1]} = e_3(e_2)^2$).

(a) Show that $e_\lambda = x^\lambda + lower \ terms$.

(b) Show that the e_λ form a basis for the symmetric polynomials in k variables.

(c) Prove the 'fundamental theorem of symmetric functions': The symmetric polynomials over $\mathbf{Z}$ are isomorphic to the polynomial ring $\mathbf{Z}[y_1, \ ..., \ y_k]$ in k variables. (*Hint* : Show that the map $y_i \mapsto e_i$ induces this isomorphism).