1. Prove that
\[ s_\lambda(x_1, x_2, \ldots, x_k) = \sum_{\mu} s_\mu(x_1, \ldots, x_{k-1}) x_k^{\lambda - |\mu|}, \]
here |\(\lambda|\) is the number of boxes of \(\lambda\) and the summation goes over all Young diagrams \(\mu\) satisfying \(\lambda_{i+1} \leq \mu_i \leq \lambda_i\), for \(i = 1, 2, \ldots, k - 1\). Here are a number of hopefully useful hints:
(a) Show that \(s_\lambda(x_1, x_2, \ldots, x_k) = (x_1 x_2 \ldots x_k)^{\lambda_k} s_\lambda\), where \(\lambda_i = \lambda_i - \lambda_k\).
(b) If \(\lambda_k = 0\), show that \(A_\lambda = \det(x_j^{\ell_i} - x_k^{\ell_i})\), with \(\ell_i = \lambda_i + k - i\) and with \(1 \leq i, j \leq k - 1\).
\textit{Remark:} This formula is useful for finding a combinatorial interpretation of Kostka numbers and for restriction rules of representations of \(Gl(k)\) to \(Gl(k - 1)\).

2. (a) Using the restriction formula for representations of \(S_n\), show that the dimension of the irreducible \(S_n\)-module \(V_\lambda\) is equal to the number of standard tableaux of shape \(\lambda\).
(b) The Bratteli diagram for the algebras \(CS_n, n \in \mathbb{N}\) is given as follows: Draw all Young diagrams with \(n\) boxes in the \(n\)-th line, and connect a diagram \(\mu\) on line \(n - 1\) with a diagram \(\lambda\) on line \(n\) if and only if \(\mu \subseteq \lambda\). Show that the dimension of \(V_\lambda\) is equal to the number of paths from \([1]\) to \(\lambda\).
(c) Calculate the number of all paths of length \(2n\) going from \([1]\) down to the \(n\)-th line and back to \([1]\), not necessarily the same way (each connecting line segment between two Young diagrams as in (b) has length 1).

3. Let \(d_\lambda = \text{dim } V_\lambda\), where \(V_\lambda\) is the irreducible representation of \(S_n\) corresponding to the Young diagram \(\lambda\). What is \(\sum_{\mu \supset \lambda} d_\mu\), where the summation goes over all Young diagrams \(\mu\) obtained by adding a box to \(\lambda\). \textit{Hint:} For my favorite solution, consider induced representations.

4. In the following we consider symmetric polynomials in \(k\) variables with all Young diagrams having at most \(k\) rows (we set \(s_\lambda = 0\) if \(\lambda\) has more than \(k\) rows). We have already seen that the Schur function \(s_{[r]}\) is equal to the \(r\)-th elementary symmetric function
\[ e_r = \sum_{1 \leq i_1 < \ldots < i_r \leq k} x_{i_1} x_{i_2} \ldots x_{i_r}. \]
Define for any Young diagram \(\lambda\) the polynomial \(e_\lambda\) to be the product of the \(e_r\)'s corresponding to the columns of \(\lambda\) (e.g. \(e_{[3,3,1]} = e_2^3\)).
(a) Show that \(e_\lambda = x^\lambda + \text{lower terms}\).
(b) Show that the \(e_\lambda\) form a basis for the symmetric polynomials in \(k\) variables.
(c) Prove the ‘fundamental theorem of symmetric functions’: The symmetric polynomials over \(\mathbb{Z}\) are isomorphic to the polynomial ring \(\mathbb{Z}[y_1, \ldots, y_k]\) in \(k\) variables. \textit{(Hint:} Show that the map \(y_k \mapsto e_i\) induces this isomorphism).