EXERCISES MATH 202C - 3rd Assignment

Do problem 4 of the last assignment if you have not done so already.

1. (a) Calculate all Schur functions with $k = 3$ variables belonging to Young diagrams with ≤ 4 boxes using $s_{[1^r]} = e_r$, the r-th elementary symmetric function and the Pieri rule, by induction. You may also use $s_{[r]} = h_r$, the r-th homogeneous symmetric function, to save work - although your algorithm should not depend on it.
(b) Write down an inductive algorithm for calculating Schur functions using the Pieri rule.

2. (a) Let $\dim V = 3$. Describe the decomposition of $V^{\otimes 4}$ into irreducible $Gl(3)$-modules and into irreducible S_4-modules (with multiplicities).
(b) Let $W_\lambda = e_\lambda V^{\otimes n}$, where e_λ is a minimal idempotent in $\mathbf{C}S_n$ corresponding to a Young diagram λ with n boxes, and $\dim V > 10$. Calculate the decomposition of the $Gl(V)$-module $W_{[5,3,3,2]} \otimes W_{[1,1,1]}$ into a direct sum of irreducible $Gl(V)$-modules.

3. Let $e_r^{(m)}$ be the elementary symmetric function of degree r in variables $x_1, x_2, \ldots$ $x_{m-1}, x_{m+1}, \ldots\ x_k$, for $1 \leq m \leq k$. Let $K = ((-1)^{k-i} e_{k-i}^{(m)})_{1\leq i,m\leq k}$. Let $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbf{N}^k$, and let the matrices A_α, H_α be defined by $A_\alpha = (x_j^{\alpha_i})$, $H_\alpha = (h_{\alpha_i - k + j})$ (with $h_r = 0$ for $r < 0$ and $h_0 = 1$). Show that $A_\alpha = H_\alpha K$. Here are a few hints:
(a) Show that $\prod_{i=1}^k (1 + x_i t) = \sum_r e_r t^r$, with the e_r being the elementary symmetric functions.
(b) Let $E^{(m)}(t) = \sum_{r=0}^{k-1} e_r^{(m)} t^r$ and let $H(t) = \prod_i (1 - x_i t)^{-1} = \sum_{r=0}^{\infty} h_r t^r$. Calculate $H(t)E^{(m)}(-t)$ in two different ways to prove the claim.

4. Prove the formula $s_\lambda = \det(h_{\lambda_i - i + j})$, with the number of rows and columns of the matrix equal to the number of rows of λ. (*Hint*: Calculate $\det H_\rho$ to find a connection between $\det K$ and $\det A_\rho$, with notations as in Problem 3 and $\rho = (k - i)_i$, as usual).