EXERCISES MATH 202C - 4th Assignment

1. Calculate the multiplicity of the S_{20} module $V_{[6,5,4,3,2]}$ in the representation induced from the $S_{14} \times S_6$ module $V_{[5,4,3,2]} \otimes V_{[3,2,1]}$.

2. Let λ be a Young diagram with $\leq k$ rows, and let $H_\lambda = \prod_{i=1}^k H_{\lambda_i}$, where $H_n = \sum_{|\alpha|=n} x^\alpha$ in the variables x_1, ... x_k. Show that $(H_\lambda)_{\lambda \leq k \text{ rows}}$ forms a basis of the symmetric functions. Moreover, show that any monomial symmetric function and any Schur function is a linear combination with *integer* coefficients of the H_λs.

3. Let $f = xy^2 - x$, $f_1 = xy + 1$ and $f_2 = y^2 - 1$.
   (a) Do division with remainder, first dividing f by f_1 and then by f_2.
   (b) Do the same with f_1 and f_2 interchanged. Compare.

4. (a) Let f be an irreducible polynomial over k in one variable x (i.e. f can not be written as $f = gh$ with both g and h polynomials of degree ≥ 1). Show that if $\tilde{f} \notin \langle f \rangle$, the ideal generated by f, then $\langle f, \tilde{f} \rangle = k[x]$.
   (b) Let now $f \in k[x]$ be arbitrary, and let
   $$\langle f \rangle \subset I_1 \subset I_2 \subset ...$$
   be a sequence of ideals, with strict inequalities. Show that this sequence has to be finite.

5. Let $A = \bigoplus_\lambda A_\lambda$ be a semsimple algebra (say over the complex numbers $\mathbf{C}$), with A_λ being isomorphic to the algebra of $d_\lambda \times d_\lambda$ matrices with $d_\lambda \in \mathbf{N}$. Let $e \in A$ be an idempotent, and let $r_\lambda = Tr_{V_\lambda}(e)$, where V_λ is a simple A_λ-module. The vector $\vec{r} = (r_\lambda)$ is called the rank vector of e.

(a) Let V be an A-module, with multiplicity vector $\vec{m} = (m_\lambda)$; i.e. m_λ is the number of modules V_i isomorphic to V_λ in the decomposition of $V = \oplus_i V_i$ of simple A-modules. What is the rank of the projection representing e on V, i.e. the dimension of the subspace $e.V$?

(b) Let $e \in \mathbf{C}S_4$ with rank vector $\vec{r} = (r_\lambda)$, where $r_\lambda = 0$ if λ has one or two rows, and $r_\lambda = 1$ if λ has more than two rows. Calculate the rank of the projection via which e acts on $V^{\otimes 4}$, with $\dim V = 3$.