EXERCISES MATH 202C - Third Assignment

1. Let $E_r^{(m)}$ be the elementary symmetric function of degree r in variables $x_1, x_2, \ldots$ $x_{m-1}, x_{m+1}, \ldots x_N$, for $1 \leq m \leq N$. Let $\mathbf{K} = ((-1)^{N-i} E_{N-i}^{(m)})_{1\leq i,m\leq N}$. Let $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbf{N}^N$, and let the matrices $\mathbf{A}_\alpha$, $\mathbf{H}_\alpha$ be defined by $\mathbf{A}_\alpha = (x_j^{\alpha_i})$, $\mathbf{H}_\alpha = (H_{\alpha_i - N + j})$ (with $H_r = 0$ for $r < 0$ and $H_0 = 1$). Show that $\mathbf{A}_\alpha = \mathbf{H}_\alpha \mathbf{K}$. Here are a few hints:
   (a) Show that $\prod_{i=1}^N (1 + x_i t) = \sum_r E_r t^r$, with the E_r being the elementary symmetric functions.
   (b) Let $E^{(m)}(t) = \prod_{i=1, i\neq m}^N (1 + x_i t) = \sum_{r=0}^{N-1} E_r^{(m)} t^r$ and let $H(t) = \prod_i (1 - x_i t)^{-1} = \sum_{r=0}^\infty H_r t^r$. Calculate $H(t)E^{(m)}(-t)$ in two different ways to prove the claim.
   (c) Calculate det $\mathbf{H}_\rho$, det $\mathbf{K}$ and det $\mathbf{A}_\rho$, with $\rho = (N - i)_i$, as usual

2. (a) Prove the formula $S_\lambda = \det(H_{\lambda_i - i + j})$, with the number of rows and columns of the matrix equal to the number of rows of λ.
   (b) Let ω be the automorphism of symmetric polynomials in variables $x_1, \ldots, x_N$ defined by $\omega(E_r) = H_r$, $1 \leq r \leq N$. Calculate $\omega(S_\lambda)$. (*Hint*: Use the Pieri formula and induction).
   (c) Find a formula which expresses the Schur function S_λ via a determinant in the elementary symmetric functions E_r similar to the one in (a) and prove it.

   Remark: The formulas in (a) and (c) are usually called Jacobi-Trudi formulas. Schur functions were first discovered by Jacobi, while Schur recognized their importance for group representations.

3. Let $f = xy^2 - x$, $f_1 = xy + 1$ and $f_2 = y^2 - 1$.
   (a) Do division with remainder, first dividing f by f_1 and then by f_2.
   (b) Do the same with f_1 and f_2 interchanged. Compare.

4. (a) Let f be an irreducible polynomial over the field k in one variable x (i.e. f can not be written as $f = gh$ with both g and h polynomials of degree ≥ 1). Show that if $\tilde{f} \notin \langle f \rangle$, the ideal generated by f, then $\langle f, \tilde{f} \rangle = k[x]$.
   (b) Let now $f \in k[x]$ be arbitrary, and let

   $$\langle f \rangle \subset I_1 \subset I_2 \subset \ldots$$

   be a sequence of ideals, with strict inequalities. Show that this sequence has to be finite.

5. Calculate all tableaux of shape [3,1,1] in the letters 1,2,3 (i.e. fillings which are strictly increasing in columns and weakly increasing in rows). Check this with the dimension formula for the $Gl(3)$-module $V_{[3,1,1]}$ and with its decomposition as a $Gl(2)$-module.