EXERCISES MATH 202C - Fourth Assignment

1. (a) Let I be a monomial ideal in $k[x_1, \ ..., \ x_n]$. Let $S_1 = \{x^{\alpha(i)}, 1 \le i \le r\}$ and $S_2 = \{x^{\beta(j)}, 1 \le j \le s\}$ be two minimal generating sets (i.e. if you remove one element from one of these sets, the ideal generated by the smaller set will be smaller). Show that $S_1 = S_2$.
   (b) Show that $x^\beta \in I$ if and only if $x^{\alpha(i)}|x^\beta$ for some $\alpha(i)$ as in (a).
   (c) Prove or disprove: There exists a number M such that any monomial ideal in $k[x, y]$ can be presented with $\le M$ generators.

2. Let f and g be polynomials in $k[x_1, \ ..., \ x_n]$. The S-polynomial of f and g is defined by
   $$S(f, g) = \frac{x^\gamma}{LT(f)} f - \frac{x^\gamma}{LT(g)} g,$$
   where x^γ is the least common multiple of the leading monomials $x^{\deg f}$ and $x^{\deg g}$, and $LT(f), LT(g)$ are the leading terms of f abd g.
   (a) Show that $S(x^\alpha f, x^\beta g) = x^\mu S(f, g)$ and calculate μ. Moreover, show that the degree of $S(x^\alpha f, x^\beta g)$ is smaller than the degree of $LCM(LT(x^\alpha f), LT(x^\beta g))$.
   (b) Find two polynomials h_1 and h_2, not multiples of each other, in the ideal generated by $\{f_1, f_2, f_3\} = \{x^4y^2 - z, x^3y^3 - 1, x^2y^4 - 2z\}$ such that h_i has nonzero remainder r_i after division with remainder with respect to f_1, f_2 and f_3, for $i = 1, 2$.

3. Let $V = \{(t, t^2, t^3), \ t \in \mathbf{R}\}$; it is usually called the twisted cubic. Show that V is a variety and that its coordinate ring $k[V]$ is isomorphic to $k[x]$. (*Hint* : Show that any $f \in k[x, y, z]$ can be written as
   $$f = h_1(y - x^2) + h_2(z - x^3) + r,$$
   for suitable $h_i \in k[x, y, z]$, $i = 1, 2$ and with $r \in k[x]$).

4. So far, we know how to write the product of Schur functions $s_\lambda s_{[1^r]}$ as a sum of Schur functions. There exists a general combinatorial rule for calculating $s_\lambda s_\mu = \sum c^\nu_{\lambda\mu} s_\nu$, the Littlewood-Richardson rule. It says the following:
   The number $c^\nu_{\lambda\mu}$ is nonzero only if $\lambda \subset \nu$. In this case, fill $\nu\backslash\lambda$ with μ_i i's, $i = 1, 2, \ ...$ as for a tableau (i.e. weakly row increasing, strictly column increasing). Read the numbers in $\nu\backslash\lambda$ from right to left, top down to get a sequence of numbers $a_1a_2a_3 \ ...$. Then $c^\nu_{\lambda\mu}$ is equal to the number of tableux of $\nu\backslash\lambda$ such that at each point of reading the sequence above, the number i will have occurred at least as often as the number $i + 1$, for all i. (E.g. 1122132 is OK, 1122321 is not).

   (a) Show that this rule is compatible with the Pieri rules.
   (b) Prove it for $\mu = [21^r]$ (i.e. $\mu_1 = 2$, $\mu_i = 1$ for $2 \le i \le r+1$). What is the maximum possible value for $c^\nu_{\lambda[21^r]}$?
   (c) Outline a proof of the Littlewood Richardson rule for μ having two columns, and for μ being a hook diagram.
   (d) Calculate $c^\nu_{\lambda\mu}$ for $\lambda = [4, 2, 1]$, $\mu = [3, 2]$ and $\nu = [5, 5, 2]$.