This part of the Applied Algebra exam will be scaled to make up 60% of the whole exam. The problems have the same value, except for the last one which will count more. Try to do as many problems as possible.

1. Let \( p_n = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \) and let \( d \) be the diagonal matrix with diagonal entries \( x_1, \ldots, x_N \), where \( V = \mathbb{C}^N \). The matrix \( d \) acts on each factor of \( V^\otimes n \), thereby defining a linear action on \( V^\otimes n \). The action of \( S_n \) on \( V^\otimes n \) is given via permuting the tensor factors.
   (a) Calculate \( Tr_{V^\otimes n}(p_n d) \).
   (b) The value of \( Tr_{V^\otimes 10}(p_4 \otimes p_4 \otimes p_2 d) \) can be written as a linear combination of Schur functions. Calculate the coefficient of \( s_{[6,3,1]} \).
   (c) Calculate the multiplicity of the simple \( S_{10} \) module \( S^{[3,3,3,1]} \) in \( V^\otimes 10 \) for \( \dim(V) = N = 5 \) and for \( N = 3 \).

2. Let \( e_r \) be the \( r \)-th elementary symmetric function in the variables \( x_1, x_2, \ldots, x_n \).
   (a) Show that the determinant of \( (\partial e_i/\partial x_j)_{1 \leq i, j \leq n} \) is a homogeneous polynomial and calculate its degree.
   (b) Calculate the determinant.

3. Let \( \rho : G \to GL(V) \) be a representation of the finite group \( G \) into the group \( GL(V) \) of invertible linear maps on the vector space \( V \).
   (a) Show that also the map \( \hat{\rho} : g \in \hat{G} \mapsto \rho(g^{-1})^t \) defines a representation, where \( t \) means the transpose of a matrix.
   (b) Let \( \chi_\rho \) and \( \chi_{\hat{\rho}} \) be the characters of \( \rho \) and \( \hat{\rho} \). Show that \( \chi_{\hat{\rho}}(g) = \bar{\chi}_\rho(g) \) (i.e. the complex conjugate) for all \( g \in \hat{G} \).
   (c) Let \( V \) be a simple \( G \)-module. Show: If \( W \) is a simple \( G \) module such that the trivial representation occurs in \( V \otimes W \), then \( W \) must be isomorphic to the representation defined in (a).

4. Let \( f_1 = x^2 y^2 - x \) and \( f_2 = x^3 y - 1 \).
   (a) Calculate a Gröbner basis for \( \langle f_1, f_2 \rangle \cap k[x] \) and for \( \langle f_1, f_2 \rangle \), where \( \langle f_1, f_2 \rangle \) is the ideal in \( k[x, y] \) generated by \( f_1 \) and \( f_2 \).
   (b) What is the variety \( V(f_1, f_2) = \{(a, b) \in \mathbb{C}^2 \mid f(a, b) = 0, f \in \langle f_1, f_2 \rangle \} \) for \( k = \mathbb{C} \)?

5. Let \( G \) be the subgroup of \( S_4 \) generated by the permutations (12) and (34), and let \( V \) be the simple representation of \( S_4 \) labeled by the Young diagram [2,1,1].
   (a) Calculate the Molien series of \( k[x_1, x_2, x_3]^G \).
   (b) Let \( \tilde{G} \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \) with generators \( g_1 \) and \( g_2 \), and let \( W \) be the three-dimensional \( \tilde{G} \)-module with basis \( w_1, w_2, w_3 \) such that the action of \( G \) is given by
   \[
   g_1 w_i = \begin{cases} 
   -w_i & \text{if } i=1,2, \\
   w_3 & \text{if } i=3,
   \end{cases} \quad g_2 w_i = \begin{cases} 
   w_1 & \text{if } i=1, \\
   -w_i & \text{if } i=2,3.
   \end{cases}
   \]
   Write down a Hironaka decomposition for \( k[y_1, y_2, y_3]^\tilde{G} \).
   (c) Find presentations of the rings \( k[y_1, y_2, y_3]^\tilde{G} \) and \( k[x_1, x_2, x_3]^G \) via generators and relations.