Remark: Let $V, W$ be $G$-modules.

$\implies \text{Hom}_G(V, W)$ is a $G$-module with $G$-action defined by $g \cdot T = \text{sw}(g) T \text{sw}(g)^{-1}$ where $\text{sw}(g): w \mapsto g \cdot w$.

If $G$ is a group, $x \in G$.

$$x \cdot T = \frac{d}{d x} \left( e^{x} (b x) T e^{-x} (b x) \right)_{x=0}$$

$$= xT - TX$$

Let $\text{Hom}_G(V, W) = \{ T: V \to W, xT = T(x)W \} \quad \forall x \in G, x \cdot W$.

$\implies$ action of $G$ on $\text{Hom}_G(V, W)$ defined.

**Theorem:** If $V$ is a $G$-module $\implies$ $G$-module $W = V \oplus W'$ s.t.

$V = W \oplus W'$.

**Proof:** Assume $W$ is mod. $\implies \text{dim} W \leq \text{dim} W - 1$ and $W_{n+1} = 0$.

If $G$ acts trivially on $V/W$, then 

$G_v$ acts trivially on $V/W$.

Then $G$ acts on $W'$ via nonzero scalar on $W$.

If $W'$ is eigenspace of $G$ for eigenvalue $0$.

$\implies$ $V = W \oplus W'$ s.t. $\text{dim} W = \text{dim} V - 1$ and $W_{n+1} = 0$.

**Statement:** If $\oplus_{i=0}^n W_i = W_0$ s.t. $W_0$ is a $G$-module s.t. $W_{i+1}/W_i$ simple.

Then, by induction, $\exists W'$ s.t. $V/W = W/W_0 = W/W_0$.

By $\oplus_{i=0}^n W_i = W_0$ s.t. $W_{i+1}/W_i$ simple.

$W_{i+1}/W_i = W_{i+1}/W_i$.

**Proof:** Let $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n = V/W$.

By induction, $\exists W'$ s.t.

$\implies W' = W/W_0$.

**Proposition:** $W'$ is $G$-module arbitrary.

Consider the map $\varphi: \text{Hom}_G(V, W) \to \text{Hom}_G(V, W)$.

This is a $G$-module map (action is trivial on both $\text{Hom}_G(V, W)$ and $\text{Hom}_G(V, W)$).

The image of $\varphi = \text{Hom}_G(V, W) = C/W$.

Ker $\varphi$ is submodule of $\text{Hom}_G(V, W)$ with codimension 1.

$$\varphi \circ \varphi = \text{Hom}_G(V, W) = \ker \varphi \oplus C/W,$$

$$\varphi = \ker \varphi \oplus C/W \quad \text{if and only if} \quad \varphi^2 = \varphi.$$