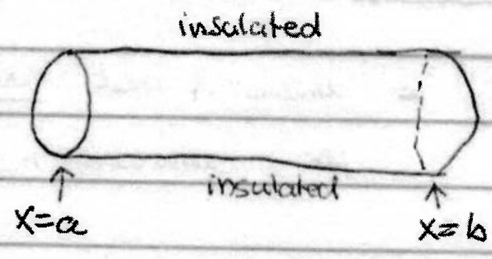
Math 110 Partial Differential Equations Lecture 1

General course information: see ~~the~~ course web page

link: math.ucsd.edu/~hwenzl/110.html

Derivation of heat equation in 1-dim case

Set-up: we consider a rod



heat exchange only possible at the 2 sides $x=a$ and $x=b$

$e(x,t)$ = heat energy density at (x,t)
= heat energy / unit volume at (x,t)

heat energy depends on

- temperature $u(x,t)$ at (x,t)
- ~~spe~~ mass density $\rho(x)$ / ~~vo~~ unit volume
- specific heat $c(x)$ = amount of energy necessary to increase one mass unit of given material by one degree

formula: $\boxed{e(x,t) = \rho(x)\, c(x)\, u(x,t)}$

heat energy at time t = $H(t) = \int_a^b e(x,t)\, \cancel{dx}\, A\, dx$

Formula 1 for change of heat energy:

$$\boxed{H'(t) = \int_a^b \frac{\partial e}{\partial t}(x,t)A\,dx}$$

Second way to calculate change of heat:

Define heat flux $\phi(x)$ = amount of heat flowing to the right at position x per time unit x area unit

Recall heat can only ~~escape~~/flow in/out at $x=a$ and $x=b$

$\Rightarrow$ $\boxed{\text{change of heat energy at time } t = \phi(a,t)A - \phi(b,t)A}$

$= \boxed{H'(t)}$

$-$ sign as $\phi(b,t)$ = amount of heat flowing to the right i.e. out of rod

Combining these two formulas, we get

$$\int_a^b \frac{\partial e}{\partial t}(x,t)A\,dx = [\phi(a,t) - \phi(b,t)]A = -\int_a^b \frac{\partial \phi}{\partial x}(x,t)A\,dx$$

- divide by A on both sides
- formula holds for all possible choices $a \le b$

$\Rightarrow$ integrated functions must coincide

$$\Rightarrow \boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}}$$

Relate both $e(x,t)$ and $\phi(x,t)$ to the temperature $u(x,t)$

have already seen: $\boxed{e(x,t) = c(x)\rho(x)\, u(x,t)}$

<u>Fourier's Law:</u>
- heat flows from hot to cold
- the greater the difference between hot and cold the faster the heat flows

$\phi \approx -\frac{\partial u}{\partial x}$

precise mathem. formulation: $\boxed{\phi(x,t) = -K_0 \frac{\partial u}{\partial x}}$

physical constant depending on material

$\Rightarrow$ plug into formula on previous page:

$$\Rightarrow \boxed{c(x)\rho(x) \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}}$$

Heat Equation

<u>Remark</u> We have done derivation without assuming any internal heat sources in the rod.
In the book the derivation is done for this more complicated case
read chapter 2.1, conservation of heat energy (exact)
(page 3 ...)

obtain: If $Q(x,t)$ = heat energy per unit volume generated per unit time

$$\Rightarrow \boxed{c(x)\rho(x) \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Q(x,t)}$$