heat equation will have source: \( Q(x,t) = \text{heat energy generated by per unit volume per unit time} \)

same derivation as before except:

\[
\frac{\partial H}{\partial t} = \int_{a}^{b} \left[ \phi(a,t) - \phi(b,t) \right] A + \int_{a}^{b} Q(x,t) A \, dx
\]

\[
\frac{\partial}{\partial t} \int e(x,t) A \, dx
\]

\[
\Rightarrow \int_{a}^{b} \frac{\partial e}{\partial t} (x,t) A \, dx = - \int_{a}^{b} \frac{\partial \phi}{\partial x} (x,t) A \, dx + \int_{a}^{b} Q(x,t) A \, dx
\]

true for all \( a < b \)

\[
\Rightarrow \frac{\partial e}{\partial t} = - \frac{\partial \phi}{\partial x} + Q
\]

\( e(x,t) = c(x) s(x)c(x,t) \)

Fourier's law \( \phi = - K_0(x) \frac{\partial u}{\partial x} \)

general case of heat equation:

\[
\int c(x) s(x) \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( K_0(x) \frac{\partial u}{\partial x} \right) + Q
\]
Special Case of

We derived the heat equation: \[ \frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2} \]

for a medium with a constant temperature \( u(x,t) \) depends on initial and

\[ u(x,0) = f(x) \]

and on boundary conditions at each end

\[ u(0,t) = T_1(t) \]

\[ u(L,t) = T_2(t) \]

Special Case: Equilibrium temp distribution

- Assume temperatures constant at boundaries

\[ u(0,t) = T_1 \]

\[ u(L,t) = T_2 \]

Equilibrium solution means solution with constant temperature

\[ \Rightarrow \frac{\partial u}{\partial t} = 0 \]

\[ \Rightarrow \text{solution of } \partial \text{ diff. equ. } \frac{\partial^2 u}{\partial x^2} = 0 \]

(u only depends on \( x \))

\[ \Rightarrow u(x) = C_1 x + C_2 \]

Example: Assume temperatures constant at boundaries

\[ u(0,t) = T_1, u(L,t) = T_2 \]

\[ T_1 = u(0) = C_1 0 + C_2 = C_2 \]

\[ T_2 = u(L) = C_1 L + C_2 \]

\[ \Rightarrow \begin{cases} C_2 = T_1 \\ T_2 = C_1 L + T_1 \end{cases} \]

Solution:

\[ u(x,t) = \frac{T_2 - T_1}{L} x + T_1 \]
Insulated boundaries

$\frac{\partial u}{\partial x}(0,t) = 0 \quad \frac{\partial u}{\partial x}(L,t) = 0 \quad \forall t$

as before, assume equilibrium state $\frac{\partial u}{\partial x} = 0$, we have

$u(x) = C_1 x + C_2$

$\Rightarrow 0 = u'(0) = C_1$

$(*)$

$0 = u'(L) = C_1$

need additional information for calculating $C_2$

e.g. from initial condition $u(x,0) = f(x) = C_2$

Equilibrium states distributions do occur as limiting cases under stable conditions.

Example: consider rod w/o heat source

perfectly insulated physics suggests: whatever initial conditions may have been, temperature distribution will eventually be constant over whole rod.
In mathematical terms:

**General Case:** If both ends of rod insulated

\[ \Rightarrow \lim_{t \to \infty} u(x, t) = C_2 \text{ constant for all } x \]

**how to determine** \( C_2 \)?

**Sot.** use initial cond.: \( u(x, 0) = f(x) \)

**heat energy** \( H(t) = \int_0^L c_0 u(x, 0) \, dx = \int_0^L c_0 f(x) \, dx \)

**rod perfectly insulated \( \Rightarrow H(t) = \text{const.} = H(0) \forall t \)**

\[ \Rightarrow \quad C_2 = \lim_{t \to \infty} u \]

\[ \int_0^L c_0 f(x) \, dx = H(t) = \lim_{t \to \infty} H(t) = \lim_{t \to \infty} \int_0^L c_0 u(x, t) \, dx \]

\[ \Rightarrow \quad C_2 = \frac{1}{L} \int_0^L f(x) \, dx \]

**Result:** For a perfectly insulated rod, the temperature \( u(x, t) \) will go to the average temperature \( \frac{1}{L} \int_0^L f(x) \, dx \) at time \( t = 0 \).