Math 102
Winter '08
Homework #4

$\boxed{2.4}$

② $A = \begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{pmatrix}$ $\quad u = \begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

I. Col A: $\begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$ Col A has basis $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

$\dim(\text{Col} A) = 1$

Nul A: $\left(\begin{array}{cccc|c} 0 & 1 & 4 & 0 & 0 \\ 0 & 2 & 8 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 + 4x_4 = 0$
$\Rightarrow x_1 = -4x_4$

$\therefore$ Nul $A = \left\{ \begin{pmatrix} -4x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \right\} = \left\{ x_4 \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} : x_2, x_3, x_4 \in \mathbb{R} \right\}$

So Nul A has basis
$\left\{ \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \Rightarrow \dim(\text{Nul} A) = 3$.

Col(A^T): $\begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 4 & 8 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$ Col(A^T) has basis $\left\{ \begin{pmatrix} 0 \\ 1 \\ 4 \\ 0 \end{pmatrix} \right\}$

$\Rightarrow \dim(\text{Col}(A^T)) = 1$.

Nul(A^T): $\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 4 & 8 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 + 2x_2 = 0$
$x_1 = -2x_2$

So Nul$(A^T) = \left\{ \begin{pmatrix} -2x_2 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \right\} = \left\{ x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} ; x_2 \in \mathbb{R} \right\}$

$\therefore$ Nul(A^T) has basis $\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\} \Rightarrow \dim(\text{Nul} A^T) = 1$.

II.

Col U: $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$ Basis for Col U is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$

$\Rightarrow \dim(\text{Col } U) = 2.$

Nul U $\left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 + 2x_2 + x_4 = 0 \\ x_2 + x_3 = 0 \end{array} \Rightarrow \begin{array}{l} x_2 = -x_3 \\ x_1 - 2x_3 + x_4 = 0 \end{array}$

$\Rightarrow x_1 = 2x_3 - x_4$

$\text{Nul } U = \left\{ \begin{pmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \right\} = \left\{ x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ s.t. } x_3, x_4 \in \mathbb{R} \right\}$

So basis for Nul U is $\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow \dim(\text{Nul } U) = 2.$

Col(U^T): $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow$ basis for Col(U^T) is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \Rightarrow \dim(\text{Col } U^T) = 2.$

Nul(U^T): $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 \text{ is free} \end{array}$

$\therefore \text{Nul}(U^T) = \left\{ \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \right\} = \left\{ x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ s.t. } x_3 \in \mathbb{R} \right\}$

So Nul(U^T) has basis $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow \dim(\text{Nul } U^T) = 1$

<u>NOTE</u>: I accidentally did computations for U matrix in #3 instead of #2. However, the matrices have very similar properties so if you were confused about U in #2, looking at this solution should help!

(5) $AB = 0$.

<u>Claim</u>: $\operatorname{col}(B) \subset \operatorname{Nul}(A)$.

<u>Pf</u>: Let $z \in \operatorname{Col} B$.

Then there exists an x such that

$$Bx = z.$$

$$\therefore Az = A(Bx) = (AB)x = 0x = 0$$

So by definition of Nullspace, $z \in \operatorname{Nul} A$.

$\Rightarrow \operatorname{col}(B) \subset \operatorname{Nul}(A)$. $\boxed{\checkmark}$

(6) A $m \times n$ with $\operatorname{rank}(A) = r$. $\quad m\left(\overset{n}{\quad}\right)$

(a) A has a 2-sided inverse if $m = n = r$

(b) $Ax = b$ has infinitely many solutions for every b if:

(i) $m < n$ <u>and</u>

(ii) cols of A span $\mathbb{R}^m$, i.e. $r = m$.

(8) $\overset{m \times n}{A}x = 0$ has only the zero solution

$\Rightarrow \operatorname{Rank}(A) = \min\{m, n\}$

$\Rightarrow$ cols of A are linearly independent.

(27) A is $m \times n$, $\operatorname{rank}(A) = r$. There are right-hand sides b for which $Ax = b$ has <u>no</u> solution.

$m\left(\overset{n}{\quad}\right)$

(a) $r < m$ since columns of A don't span $\mathbb{R}^m$

$r \le n$ since rank can never exceed # of cols.

(b) $A^T y = 0$ has a nonzero solution since A^T has m columns, and $\operatorname{rank} A^T = \operatorname{rank} A < m$ so the columns form a dependent set

$\therefore \exists$ nonzero solution to $A^T y = 0$.

(29) $\underset{3\times 4}{A} = \underset{3\times3}{\begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{pmatrix}} \underset{3\times4}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}}$ $\quad$ $\text{Nul } A \subseteq \mathbb{R}^4$, $\text{Col } A \subseteq \mathbb{R}^3$

Col A has dim 3 and lives in $\mathbb{R}^3 \Rightarrow \text{Col } A = \mathbb{R}^3$

$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is basis for Col A.

$\text{Col}(A^T) = \text{Row } A$ has basis $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}$.

$\text{Nul}(A^T)$: A^T $4\times3 \Rightarrow \overset{3}{\overset{\shortparallel}{\text{Rk}(A^T)}} + \dim \text{Nul}(A^T) = 3 \Rightarrow \text{Nul } A^T = \{0\}$

Nul A: $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$ $\quad$ $x_1 = -2x_4 - 3(-2x_4) - 4x_4 = 0$

$x_2 + 2x_3 + 3x_4 = 0 \Rightarrow x_2 = -2x_3 - 3x_4 = x_4$

$x_3 + 2x_4 = 0$ $\quad$ $x_3 = -2x_4$

So $\text{Nul } A = \left\{ \begin{pmatrix} 0 \\ x_4 \\ -2x_4 \\ x_4 \end{pmatrix} \right\} = \left\{ x_4 \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix} \right\} \Rightarrow$ Nul A has basis $\left\{ \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

(32) $\text{Col } A = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$\text{Col}(A^T) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\text{Nul}(A)$: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_2 = 0 \\ x_3 = 0 \\ x_1 = \text{free} \end{array}$ $\therefore \text{Nul}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

$\text{Nul}(A^T)$: $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = \text{free} \end{array}$ $\therefore \text{Nul}(A^T) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\text{Col}(I+A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\text{Col}(I+A)^T = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\text{Nul}(I+A)$: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{array}{l} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \\ x_3 = 0 \end{array} \Rightarrow \text{Nul}(I+A) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\text{Nul}(I+A)^T = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

37 a) True since rank A = rank A^T.

b) False $A = (1 \ 0)$.

$Nul(A^T) = Nul\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \{(0)\}$.

$Nul((A^T)^T) = Nul((1 \ 0)) = span\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\} \neq \{(0)\}$.

c) False: As long as A is square & invertible

$Rowsp(A) = Col(A)$

but this does not imply that A is symmetric (i.e. $A = A^T$).

ex: $A = \begin{pmatrix}1 & 2\\3 & -7\end{pmatrix} \Rightarrow Row(A) = Col(A) = \mathbb{R}^2$

but $A \neq A^T$.

d) True: $Row(A) = Col(A^T) = Col(A)$

$\Downarrow$

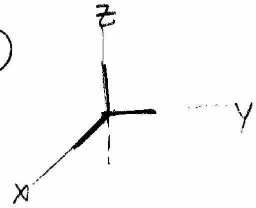
since $A^T = -A$

2.6

(i) (A) $T\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $T\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $T\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

so $M_T=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$

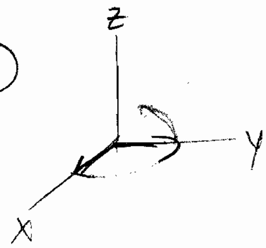
(B)



$T\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $T\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $T\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\-1\end{pmatrix}$

$\therefore$ $M_T=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$

(C)



$T\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $T\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\0\\0\end{pmatrix}$, $T\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$

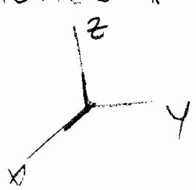
$\therefore$ $M_T=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}$

(D) composition of transformations corresponds to matrix multiplication!

Break down T into 3 steps

- A rotates x-y plane through 90°: $M_A=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}$

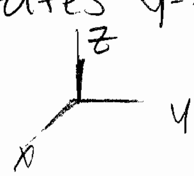
- B rotates x-z plane through 90°: $M_B=\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix}$



$B\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\-1\end{pmatrix}$, $B\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$

$B\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$

- C rotates y-z plane through 90°: $M_C=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}$



$C\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $C\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$

$C=\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\-1\\0\end{pmatrix}$

$\therefore M_T=M_CM_BM_A=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix}\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$

$=\boxed{\begin{pmatrix}0&0&1\\0&-1&0\\1&0&0\end{pmatrix}}$

2.6

(7) $\frac{d}{dt}(1)=0$, $\frac{d^2}{dt^2}(0)=0$

$\frac{d}{dt}(t)=1$, $\frac{d^2}{dt^2}(t)=0$

$\frac{d}{dt}(t^2)=2t$, $\frac{d^2}{dt^2}(t^2)=2=2(1)$

$\frac{d}{dt}(t^3)=3t^2$, $\frac{d^2}{dt^2}(t^3)=6t=6(t)$

$$M_T=\begin{pmatrix}0&0&2&0\\0&0&0&6\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$\underline{\text{Col}(M_T)}=\text{span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right\}$

which means that the image space of the transformation is $sp\{1,t\}$

$\underline{\text{Nul}(M_T)}$: $x_3=0$
$x_4=0$
x_1, x_2 free.

$\therefore \text{Nul}(M_T)=sp\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right\}$

which means that polynomials in $sp\{1,t\}$ get sent to <u>zero</u> under this transformation.

(8) $1(2+3t)=2+3t=2(1)+3(t)$

$t(2+3t)=2t+3t^2=2(t)+3(t^2)$

$t^2(2+3t)=2t^2+3t^3=2(t^2)+3(t^3)$

$t^3(2+3t)=2t^3+3t^4=2(t^3)+3(t^4)$

$$\therefore M_T=\begin{pmatrix}2&0&0&0\\3&2&0&0\\0&3&2&0\\0&0&3&2\\0&0&0&3\end{pmatrix}$$

(9) $\frac{d^2u}{dt^2} = u$ has solutions: $u_1(t) = e^t$
$u_2(t) = e^{-t}$

since $\frac{d^2}{dt^2}(e^{-t}) = \frac{d}{dt}(-e^{-t}) = e^{-t}$.

u_1 and u_2 are independent:
$c_1 e^t + c_2 e^{-t} = 0$ <u>for all</u> t
only if $c_1 = c_2 = 0$.

(16) $\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \mapsto \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ $A = \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}$

$\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \mapsto \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

$\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \mapsto \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

A^2 has the effect of shifting the components of a vector twice:
$(x_1, x_2, x_3, x_4) \Rightarrow (x_2, x_3, x_4, x_1) \Rightarrow (x_3, x_4, x_1, x_2)$

A^3 shifts components $\underline{3}$ times, so that (x_1, x_2, x_3, x_4) becomes (x_4, x_1, x_2, x_3)

so that applying A one more time gets us back to the original vector
$\Rightarrow A^3 = A^{-1}$

(This could also be verified via matrix multiplication)

(18) $S = \{p(x) \in \mathbb{P}_3 \mid \int_0^1 p(x)\,dx = 0\}$

I. Verify S is a subspace:

(i) Let $p, q \in S$. Then $\int_0^1 (p(x)+q(x))\,dx$

$$= \int_0^1 p(x)\,dx + \int_0^1 q(x)\,dx$$

$$= 0 + 0 \quad \text{since } p, q \in S$$

$$= 0$$

$\therefore\ p+q \in S$. ✓

(ii) Let $p \in S$, $\alpha \in \mathbb{R}$. Then $\int_0^1 \alpha p(x)\,dx$

$$= \alpha \int_0^1 p(x)\,dx$$

$$= \alpha \cdot 0 \quad \text{since } p \in S$$

$$= 0$$

$\therefore\ \alpha p \in S$ ✓

(iii) $0 \in S$ since $\int_0^1 0\,dx = 0$. ✓

II. Find a basis for S

Let $T: \mathbb{P}_3 \Rightarrow \mathbb{R}$ be the integration operator. We find a matrix representation A for T. We then find a basis for its nullspace!

$$\left.\begin{aligned} T(1) &= \int_0^1 1\,dt = t\Big|_0^1 = 1 \\ T(t) &= \int_0^1 t\,dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2} \\ T(t^2) &= \int_0^1 t^2\,dt = \frac{t^3}{3}\Big|_0^1 = \frac{1}{3} \\ T(t^3) &= \int_0^1 t^3\,dt = \frac{t^4}{4}\Big|_0^1 = \frac{1}{4} \end{aligned}\right\} \Rightarrow \underset{1\times 4}{T} = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \end{pmatrix}$$

Null T: $(1\ \ 1/2\ \ 1/3\ \ 1/4 \mid 0) \Rightarrow x_1 = -\frac{1}{2}x_2 - \frac{1}{3}x_3 - \frac{1}{4}x_4$

$$\therefore\ \text{Null}\,T = \left\{ \begin{pmatrix} -\frac{1}{2}x_2 - \frac{1}{3}x_3 - \frac{1}{4}x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ s.t. } x_2, x_3, x_4 \in \mathbb{R} \right\} =$$

$\longrightarrow$

$$= \left\{ x_2\begin{pmatrix}-1/2\\1\\0\\0\end{pmatrix} + x_3\begin{pmatrix}-1/3\\0\\1\\0\end{pmatrix} + x_4\begin{pmatrix}-1/4\\0\\0\\1\end{pmatrix} \text{ s.t. } x_2, x_3, x_4 \in \mathbb{R} \right\}. \quad (*)$$

Note $S = \ker(T)$

$\therefore$ Basis for S = Basis for $\ker T$

From $(*)$, a basis for $\ker(T)$ is

$$\boxed{\left\{-\tfrac{1}{2}+t,\ -\tfrac{1}{3}+t^2,\ -\tfrac{1}{4}+t^3\right\}}$$

(22) a) $T(v_1, v_2) = (v_2, v_1)$ $\quad v = (v_1, v_2),\ w = (w_1, w_2)$

$$\begin{aligned} T(\alpha v + \beta w) &= T(\alpha v_1 + \beta w_1, \alpha v_2 + \beta w_2) \\ &= (\alpha v_2 + \beta w_2, \alpha v_1 + \beta w_1) \\ &= \alpha(v_2, v_1) + \beta(w_2, w_1) \\ &= \alpha T(v) + \beta T(w) \end{aligned}$$

$\therefore$ T is linear.

b) $T(v_1, v_2) = (v_1, v_1)$

$$\begin{aligned} T(\alpha v + \beta w) &= T(\alpha v_1 + \beta w_1, \alpha v_2 + \beta w_2) \\ &= (\alpha v_1 + \beta w_1, \alpha v_1 + \beta w_1) \\ &= \alpha(v_1, v_1) + \beta(w_1, w_1) \\ &= \alpha T(v) + \beta T(w). \end{aligned}$$

$\therefore$ T is linear.

c) $T(v_1, v_2) = (0, v_1)$

$$\begin{aligned} T(\alpha v + \beta w) &= T(\alpha v_1 + \beta w_1, \alpha v_2 + \beta w_2) \\ &= (0, \alpha v_1 + \beta w_1) \\ &= (0, \alpha v_1) + (0, \beta w_1) \\ &= \alpha(0, v_1) + \beta(0, w_1) \\ &= \alpha T(v) + \beta T(w). \end{aligned}$$

$\therefore$ T is linear.

$\longrightarrow$

d) $T(v)=(0,1)$

$T(\alpha v+\beta w) = T(\alpha v_1+\beta w_1, \alpha v_2+\beta w_2)$
$=(0,1)$

$\alpha T(v) + \beta T(w) = \alpha(0,1) + \beta(0,1)$
$= (0, \alpha+\beta)$
$\neq (0,1)$ in general.

$\therefore$ T is <u>not</u> linear.

(33) Let $M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Then $M^T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Does there exist a matrix A such that $AM = M^T$?

If so, we'd have

$$AM = \left(A\begin{pmatrix} 0 \\ 1 \end{pmatrix} \; A\begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = M^T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow A\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which is impossible since $A\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for every A.

$\therefore$ There <u>does not</u> exist such an A.