MATH 241B FUNCTIONAL ANALYSIS - NOTES

EXAMPLES OF C* ALGEBRAS

These are informal notes which cover some of the material which is not in the course book. The main purpose is to give a number of nontrivial examples of C^* algebras. In order to prove that they are non-isomorphic, we need several results which are of interest in their own right.

1. Technical Lemmas This section contains two technical lemmas about approximating projections.

Lemma 1 If p and q are projections in a C^* algebra A such that $\|p-q\|<1$, then p and q are equivalent in A, $p\sim q$, i.e there exists an element $u\in A$ such that $uu^*=p$ and $u^*u=q$.

Proof. We have $\|p-pqp\|=\|p(p-q)p\|\le\|p\|\|p-q\|\|p\|=\|p-q\|<1$. Hence pqp is invertible in the C^* algebra pAp with inverse $\sum_{i=0}^\infty(p-pqp)^i$ (where $(p-pqp)^0=p$, the identity of pAp). As pqp is positive, there exists the positive element $x=(pqp)^{1/2}\in pAp$. Let $u=x^{-1}pq$. Then we have

$$uu^*=x^{-1}pqpx^{-1}=x^{-1}x^2x^{-1}=p.$$

Using $p=x^{-2}pqp=x^{-2}qp$ (as $xp=x\in pAp$), we obtain

$$(qx^{-2}q)qpq=q(x^{-2}qp)q=qpq.$$

By the same arguments as before, we can show that qpq is invertible in qAq. Multiplying the equation above by $(qpq)^{-1}$ from the right, we obtain

$$q=qx^{-2}q=(qpx^{-1})(x^{-1}pq)=u^*u.$$

Lemma 2 Let $p\in A$ be a projection. Then there exists an $\varepsilon>0$ such that if $\|b-p\|<\varepsilon$ for a selfadjoint $b\in A$ then there exists a projection $q\in C^*(b)$ such that $q\sim p$.

Proof. First observe that $\|b\|=\|b-p+p\|\le\|b-p\|+\|p\|\le 1+\varepsilon$. Hence we obtain

$$\|b^2-b\|=\|b^2-bp+bp-p^2+p-b\|\le\|b\|\|b-p\|=\|b-p\|\|p\|+\|p-b\|\le(3+\varepsilon)\varepsilon.$$

This implies that $\lambda(1-\lambda)|<4\varepsilon$ for any $\lambda\in\sigma(b)$. For ε sufficiently small (say $\varepsilon=1/20$), this implies that $|\lambda|<1/3$ or $|1-\lambda|<1/3$. Let χ be any continuous function on the real line such that $\chi(\lambda)=0$ for $|\lambda|<1/3$ and $\chi(\lambda)=1$ for $|1-\lambda|<1/3$. Then it follows from spectral calculus that $q=\chi(b)$ is a projection with $\|q-b\|=\|\chi(\lambda)-\lambda\|\le 1/3$, where $\|\chi(\lambda)-\lambda\|$ is the supremum of $|\chi(\lambda)-\lambda|$ on $\sigma(b)$. It follows that

$$\|q-p\|\le\|q-b\|+\|b-p\|\le 1/3+\varepsilon<1.$$

2. Some non-isomorphic ***AF*** **algebras** A C^* algebra A is called an AF (almost finite) algebra if there exists a sequence of finite dimensional C^* algebras

$$A_1\subset A_2\subset A_3\subset\ \ldots$$

such that the union $\bigcup A_n$ is dense in A.

Theorem 1 (a) Any projection in an AF algebra A is equivalent to a projection in a finite dimensional algebra A_n for some n.

(b) Let A and B be AF algebras with unique traces. If there exists a projection $p\in A_n$ for some n such that $tr_A(p)\neq tr_B(q)$ for all projections $q\in B_n$, $n\in\mathbf{N}$, then A is not isomorphic to B.

Proof. If p is a projection in A and $\varepsilon > 0$, we can find an element b in some A_n such that $\|b - p\| < \varepsilon$. As also $\|b^* - p\| < \varepsilon$, it follows that $\|Re(b) - p\| < \varepsilon$, where $Re(b) = \frac{1}{2}(b + b^*)$ is self-adjoint. Then claim (a) follows from the previous lemma after choosing ε sufficiently small.

For part (b), if there were an isomorphism $\Phi : A \to B$, it would induce a trace functional tr' on A from the trace tr_B on B by $tr'(a) = tr_B(\Phi(a))$. By uniqueness of the traces, tr' would coincide with the trace tr_A on A. But then $tr_B(\Phi(p)) = tr_A(p)$ for the projection $\Phi(p) \in B$. But as $\Phi(p)$ would be equivalent to some projection q in some B_n, by (a), this would contradict our assumption.

Corollary 1 Let $A = \bigotimes_{i=1}^{\infty} M_k$ and $B = \bigotimes_{i=1}^{\infty} M_m$. If there exists a prime number p_o such that $p_o | k$, but $p_o \not| m$, then A is not isomorphic to B.

Proof. If tr is a normalized trace on A (normalized means that $tr(1) = 1$), then the restriction of tr to the finite dimensional subalgebra $A_n = \bigotimes_{i=1}^{n} M_k \cong M_{k^n}$ is the unique normalized trace on M_{k^n}, i.e. the usual sum of diagonal elements divided by k^n (see exercise below). So any two traces on A coincide on all of the finite dimensional subalgebras A_n, and hence they coincide on all of A by continuity. Similarly, the trace on B is unique. As $p_o | k$, we have a projection p in M_k, and hence also in A such that $tr(p) = 1/p_o$. On the other hand, if q is a projection in B_n, $tr(q) = j/m^n$ for some $j \in \mathbf{N}$. As $p_o \not| m$, $tr(q) \neq tr(p)$. Hence $A \ncong B$, by the previous Theorem.

Exercise : Show that any functional ϕ on M_n satisfying $\phi(ab) = \phi(ba)$ is equal to a multiple of the usual trace Tr, given by the sum of the diagonal entries. This can be done by calculating directly $\phi(E_{ij})$, where the E_{ij} are the usual matrix units. E.g. we have

$$\phi(E_{ij}) = \phi(E_{ii}E_{ij}) = \phi(E_{ij}E_{ii}) = 0, \quad \text{if } i \neq j.$$

The algebras in the theorem belong to one of the simplest classes of AF algebras, also called UHF algebras (uniformly hyperfinite algebras). In order to study more complicated AF algebras, we will need some simple statements about the representation theory of C^* algebras. We say that a Hilbert space H is an A-module for the C^* algebra A if $a \in A$ acts via a bounded linear operator $\rho(a)$ on H such that $\rho(a^*) = \rho(a)^*$, the usual adjoint of $\rho(a) \in B(H)$. In order to keep the notation simple, we may sometimes just deal with the case that A is a closed subalgebra of $B(H)$. In view of the GNS construction, this is not an essential restriction. An A module H is called *simple* if the only submodules of H are H itself and 0.

Lemma 3 Let $A \subset B(H)$ be a C^* algebra. Then

(a) If $H_1 \subset H$ is an A-submodules of H, we have the decomposition $H = H_1 \oplus H_1^{\perp}$ as A-submodules.

(b) If A is a finite dimensional C^* algebra, any finite dimensional A module is a direct sum of simple A-modules.

Proof. For part (a) we only need to show that if $\xi \in H_1^{\perp}$ and $a \in A$, then also $a\xi \in H_1^{\perp}$. This follows from

$$(a\xi, \eta) = (\xi, a^*\eta) = 0 \quad \text{for } \eta \in H_1,$$

as $a^*\eta \in H_1$. Part (b) follows by induction on the dimension on H, using part (a) in case H is not simple.

Inclusions of finite-dimensional C^* algebras We assume that we have an inclusion of finite dimensional C^* algebras $A \subset B$ such that both A and B have the same identity element 1. We assume

$$A \cong \bigoplus_i M_{d_i} \subset B \cong \bigoplus_j M_{e_j}.$$

Let $V_i \cong \mathbf{C}^{d_i}$ and $W_j \cong \mathbf{C}^{e_j}$ be simple A- resp B-modules. Then W_j, viewed as an A-module, can be written as a direct sum of simple A-modules, by Lemma 3. Let g_{ij} be the number of those modules which are isomorphic to V_i. Comparing dimensions, we get

$$e_j = \sum_i g_{ij} d_i \quad \text{and} \quad G^t \mathbf{d} = \mathbf{e},$$

where $G = (g_{ij})$ is called the inclusion matrix for $A \subset B$, and $\mathbf{e} = (e_j)$, $\mathbf{d} = (d_i)$ are the dimension vectors of B and A respectively. We also assume B has a normalized trace tr. Then $tr_{|M_{e_j}} = w_j Tr$ for some non-negative w_j, by the exercise above. We call the vector $\mathbf{w} = (w_j)$ the weight vector of the trace tr. We can similarly define the weight vector $\mathbf{v} = (v_i)$ of the restriction of tr to A. We then also have

$$v_i = \sum_j g_{ij} w_j \quad \text{and} \quad \mathbf{v} = G\mathbf{w}.$$

3. Perron-Frobenius Theorem and more AF-algebras We first deal with a useful technical theorem, the Perron-Frobenius Theorem. To prove it, we first prove the following useful proposition.

Proposition Let K be a compact set, and let $g : K \to K$ be a continuous map. Assume that K has a metric d for which $d(g(x), g(y)) < d(x, y)$ for all $x \neq y$ in K. Then $\bigcap_n g^n(K) = \{x\}$ for a single point $x \in K$ with $g(x) = x$.

Proof. Let $K_o = \bigcap_n g^n(K)$. Then $K_o \neq \emptyset$ and, by construction, we have

$$g(K_o) = \bigcap_n g^{n+1}(K) = K_o,$$

i.e. the map g is surjective on K_o. As K_o is compact, there exist points $x, y \in K$ such that

$$d(x, y) \geq d(x', y') \quad \text{for all } x', y' \in K_o.$$

By surjectivity of g, there exist points $x_1, y_1 \in K_o$ such that $g(x_1) = x$ and $g(y_1) = y$. But if $x \neq y$, we would get

$$d(x, y) = d(g(x_1), g(y_1)) < d(x_1, y_1) \leq d(x, y),$$

where the first inequality follows from our assumption on d, and the second one from our choice of x and y. This shows that $x = y$.

Theorem (Perron-Frobenius) Let G be a $k \times k$ matrix with all of its entries being positive. Then we have

(a) There exists an eigenvector $\mathbf{v}$ of G with all of its entries being positive.

(b) The intersection $\bigcap_{n \in \mathbf{N}} G^n(\mathbf{R}_+^k)$ is equal to $\mathbf{R}_+\mathbf{v}$, with $\mathbf{v}$ as in (a). In particular, $\mathbf{v}$ is the unique eigenvector of G such that all of its coordinates are positive, up to positive scalar multiples.

Proof. Let $\tilde{K}$ be the simplex given by the vectors $\mathbf{x}$ with $x_i \geq 0$ and $\sum_i x_i = 1$. This is obviously compact and convex. We define the map $\tilde{g} : \tilde{K} \to \tilde{K}$ by

$$\tilde{g}(x) = \frac{G\mathbf{x}}{\|G\mathbf{x}\|_1},$$

where $\|\mathbf{x}\|_1 = \sum_i |x_i|$. Then $\tilde{g}$ is well-defined on $\tilde{K}$ and does indeed map into $\tilde{K}$ (check for yourself!). It follows from Brower's fixed point theorem that $\tilde{g}$ has a fixed point $\mathbf{x}_o$. This means $G\mathbf{x}_o = \|G\mathbf{x}_o\|_1 \, \mathbf{x}_o$. As all the entries of G are positive, so are all the coordinates of its eigenvector $\mathbf{x}_o$. This finishes the proof of (a).

To prove (b), we consider the simplex K consisting of all vectors $\mathbf{x}$ satisfying $x_i \geq 0$ and $\mathbf{w}^t\mathbf{x} = 1$; here $\mathbf{w}$ is an eigenvector as in (a) for the matrix G^t with eigenvalue λ. Then it follows from the definitions that the map

$$g(x) = \frac{1}{\lambda} G(x)$$

defines a continuous map from K into itself. We define the metric d on K via

$$d(\mathbf{x}, \mathbf{y}) = \sum_i w_i |x_i - y_i|.$$

Using the eigenvector property $\sum_i g_{ij} w_i = \lambda w_j$, we obtain for $\mathbf{x} \neq \mathbf{y}$

$$d(g(\mathbf{x}), g(\mathbf{y})) = \frac{1}{\lambda} \sum_i w_i |G(x-y)_i| = \frac{1}{\lambda} \sum_i w_i | \sum_j g_{ij}(x_j - y_j)|$$

$$< \frac{1}{\lambda} \sum_{i,j} g_{ij} w_i |x_j - y_j| \ = \ \sum_j w_j |x_j - y_j| = d(\mathbf{x}, \mathbf{y});$$

here the strict inequality follows from the fact that $\sum_i w_i(x_i - y_i) = \mathbf{w}^t(\mathbf{x} - \mathbf{y}) = 1 - 1 = 0$ and from $\mathbf{x} \neq \mathbf{y}$. Hence the conditions of the proposition are satisfied, and we have $\bigcap_n g^n(K) = \{\mathbf{x}_o\}$. The claim now follows from the fact that $G^n(\mathbf{R}^k_+) = \mathbf{R}_+ g^n(K)$ for all $n \in \mathbf{N}$.

Corollary The statements of the last theorem also hold for matrices G all of whose entries are non-negative, and such that G^m only has positive entries for some integer m.

Proof. Let $\mathbf{v}$ be the Perron-Frobenius eigenvector of G^n with eigenvalue μ. Then we have

$$G^m(G\mathbf{v}) = G(G^m \mathbf{v}) = \mu G \mathbf{v}.$$

Hence also $G\mathbf{v}$ is an eigenvector of G^m with only positive coordinates, i.e. it must be a multiple of $\mathbf{v}$ by uniqueness of the Perron-Frobenius eigenvector. The other statements of the theorem now follow easily.

Theorem Let A be an AF algebra for which the inclusion matrices for $A_n \subset A_{n+1}$ are all given by a constant matrix G such that G^m only has positive entries for some positive integer m. Then A has a unique positive trace, whose weight vectors for the finite dimensional algebras A_n are given by multiples of the Perron-Frobenius eigenvector of G.

Proof. Let $\mathbf{w}_n$ be the weight vector for the restriction of the trace on A_n. Then we have $\mathbf{w}_n = G^{k-n}\mathbf{w}_k$ for all positive integers k. It follows from the Perron-Frobenius theorem and its corollary, that $\mathbf{w}_n \in \bigcap_k G^k(\mathbf{R}^k_+)$ is a multiple of the Perron-Frobenius vector of G. This shows uniqueness of the trace.

On the other hand, if $\mathbf{w}_1$ is the multiple of the Perron-Frobenius vector of G such that $\sum d_i w_i = 1$, where $\mathbf{d} = (d_i)$ is the dimension vector of A_1, we define $\mathbf{w}_n = \lambda^{1-n}\mathbf{w}_1$, where λ is the eigenvalue for $\mathbf{w}_1$. Then the restriction of the trace on A_n defined by $\mathbf{w}_n$ to A_{n-1} coincides with the trace defined by $\mathbf{w}_{n-1}$. Hence we obtain a well-defined trace on $\bigcup A_n$, which extends to A by continuity.

4. Examples of C^* algebras (a) We have already seen infinitely many examples of non-isomorphic UHF algebras, all of which have a unique trace. Further examples come from AF algebras with constant inclusion matrix. E.g. if we have the inclusion matrix

$$G \ = \ \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

it has PF eigenvector $(\lambda, 1)^t$, where $\lambda = (1+\sqrt{5})/2$ is the golden ratio, with eigenvalue λ^2. If $A_1 = \mathbf{C} \oplus \mathbf{C}$, the weight vector $\mathbf{w}_1$ of the trace is given by $(\lambda, 1)^t/\lambda^2$. It follows for any projection $p \in A$ that $tr(p)$ takes a value in the semigroup generated by the elements $1/\lambda^n$, $n \in \mathbf{N}$, for any projection $p \in A$. Semigroup here means all possible sums (with repetitions) of such elements. As the traces of projections in UHF algebras are always rational numbers, it follows from Theorem 1 that this AF algebra is not isomorphic to a UHF algebra.

(b) We can define another example of an AF algebra for which the inclusion diagram is given by Pascal's triangle. Hence $A_n \cong \bigoplus A_{n,i}$ where $A_{n,i}$ is the full matrix algebra of dimension $\begin{pmatrix} n \\ i \end{pmatrix}^2$. Then it is easy to check that we obtain for each $t \in [0,1]$ a positive trace tr on A for which the weight vector for A_n is given by $(t^i(1-t)^{n-i})_{0 \leq i \leq n}$. Hence here we have an AF algebra with infinitely many different positive traces.

This AF algebra A can also be realized in a more conceptual way as a subalgebra of the UHF algebra $\bigotimes^\infty M_2$. Let d be the 2×2 diagonal matrix with eigenvalues 1 and λ, where λ is not a root of unity. We let

d act on $\bigotimes^\infty M_2$ via conjugation, i.e. $d.(\bigotimes a_i) = \bigotimes da_i d^{-1}$. Then show that A is equal to the fixed points under this action.

(c) We finally give an example of a C^* algebra which does not allow any finite trace. Let H be a Hilbert space with orthonormal basis $(\xi_j)_{j\in\mathbf{N}}$. Fix $n \in \mathbf{N}$, $n > 1$. We define partial isometries s_i, $1 \leq i \leq n$ by $s_i\xi_j = \xi_{(n-1)j+i}$. Then one can show that

$$s_i^* s_i = 1 \quad \text{and} \quad \sum_{i=1}^n s_i s_i^* = 1.$$

Let $\mathcal{O}(n)$ be the C^* algebra generated by the s_i. Then it follows from the relations above that the only positive functional $tr \in \mathcal{O}(n)^*$ satisfying $tr(ab) = tr(ba)$ for all $a, b \in \mathcal{O}(n)$ is the zero functional. Indeed, if $tr(1) = \alpha$, we get

$$\alpha = tr(1) = tr(\sum_i s_i s_i^*) = \sum_i tr(s_i^* s_i) = n\alpha.$$

Hence $\alpha = 0$. As we have $a \leq \|a\|1$ for any positive element $a \in A$, it also follows that $0 \leq tr(a) \leq \|a\|tr(1) = 0$. The general claim now is a consequence of the fact that any element a can be written as $a_1 - a_2 + i(a_3 - a_4)$, with a_i positive for $1 \leq i \leq 4$.