MATH 241B FUNCTIONAL ANALYSIS - NOTES

SPECTRAL THEOREM

We present the material in a slightly different order than it is usually done (such as e.g. in the course book). Here we prefer to start out with an abelian $C^*$-algebra $A$ (say, the algebra $C^*(a)$ generated by a normal operator $a \in B(\mathcal{H})$) and construct from it the spectral measure. This is done as follows:

We know that $A$ is isomorphic to $C(X)$, the algebra of continuous functions on a compact set $X$. So we can view our concrete algebra $A \subset B(\mathcal{H})$ as the image of a representation $\pi : C(X) \to B(\mathcal{H})$. We first show that this representation can be extended to a representation, also denoted by $\pi$ of the bounded Borel functions $B(X)$ into $B(\mathcal{H})$. We denote by $\chi_U$ the characteristic function of a measurable subset $U \subset X$, i.e. $\chi_U(x)$ is either 1 or 0, depending on whether $x \in U$ or not. Then the spectral measure $E(U)$ is the projection given by

$$E(U) = \pi(\chi_U), \quad U \subset X \text{ measurable.}$$

If $A = C^*(a)$, for $a$ a normal operator, and $X = \sigma(a)$ its spectrum, we obtain an analog of the eigenspace decomposition of a normal matrix, with the spectral spaces $E(U)\mathcal{H}$ corresponding to direct sums of eigenspaces in the finite dimensional case.

In order to work out the details, we need a few basic results. For the definition of *regular* measure, see e.g. the course book, Appendix C.10, or wikipedia.

**Theorem A** Every bounded linear functional $\phi$ in the dual of $C(X)$ corresponds to a regular Borel measure $\mu$ on $X$ such that $\phi(f) = \int_X f \, d\mu$.

**Theorem B** For any sesquilinear form $\Psi : \mathcal{H}^2 \to \mathbf{C}$ satisfying $|\Psi(\xi, \eta)| \leq C\|\xi\|\|\eta\|$ for some constant $C > 0$ there exists a unique bounded operator $a \in B(\mathcal{H})$ with $\|a\| \leq C$ such that $\Psi(\xi, \eta) = (a\xi, \eta)$ for all $\xi, \eta \in \mathcal{H}$.

**Theorem 1** Let $A \subset B(\mathcal{H})$ be an abelian $C^*$ algebra with $A \cong C(X)$, the continuous functions on a compact space $X$. Then the homomorphism $\pi : C(X) \to A \subset B(\mathcal{H})$ can be extended to a $*$-homomorphism

$$\pi : B(X) \ \to \ B(\mathcal{H}),$$

where $B(X)$ are the bounded measurable functions on $X$.

*Proof.* Let $\hat{a} \in C(X)$ be the Gelfand transform of $a \in A$, and let $\xi, \eta \in \mathcal{H}$. Then the functional $a \mapsto (a\xi, \eta)$ can be expressed via a measure $\mu_{\xi,\eta}$ by Theorem A, i.e. we have

$$(a\xi, \eta) \ = \ \int_X \hat{a} \, d\mu_{\xi,\eta} \quad \text{for all } a \in A.$$

It is easy to see that the map $\xi \times \eta \in \mathcal{H}^2 \to \mu_{\xi,\eta}$ is linear in the first and antilinear in the second coordinate (just apply the measure to $\hat{a}$ where $a \in A$). Hence, if $f \in B(X)$, we obtain the sesquilinear form $\Psi(\xi, \eta) = \int_X f \, d\mu_{\xi,\eta}$ such that $|\Psi(\xi, \eta)| \leq \|f\|_\infty \|\xi\|\|\eta\|$. Applying Theorem B, we obtain a well-defined operator $\pi(f) \in B(\mathcal{H})$ by

$$(\pi(f)\xi, \eta) \ = \ \int_X f \, d\mu_{\xi,\eta}.$$

It is clear that $\pi(\hat{a}) = a$ for $a \in A \subset B(\mathcal{H})$, and that $\pi(\hat{ab}) = ab = \pi(\hat{a})\pi(\hat{b})$. We similarly show that $\pi$ is a homomorphism for $f, g \in B(X)$ using measures as follows: As

$$\int \hat{a}\hat{b} \, d\mu_{\xi,\eta} = (ab\xi, \eta) = \int \hat{a} \, d\mu_{b\xi,\eta},$$

we have $\hat{b} \, d\mu_{\xi,\eta} = d\mu_{b\xi,\eta}$. Hence we also have

$$\int f\hat{b} \, d\mu_{\xi,\eta} = \int f \, d\mu_{b\xi,\eta} = (\pi(f)b\xi, \eta) = (b\xi, \pi(f)^*\eta) = \int \hat{b} \, d\mu_{\xi,\pi(f)^*\eta}.$$

Hence the functional $\hat{b} \mapsto \int f\hat{b}\ d\mu_{\xi,\eta}$ is given by the measure $\mu_{\xi,\pi(f)^*\eta}$. Applying these measures to the function $g \in B(X)$ we get

$$(\pi(fg)\xi,\eta) = \int fg\ d\mu_{\xi,\eta} = \int g\ d\mu_{\xi,\pi(f)^*\eta} = (\pi(g)\xi,\pi(f)^*\eta) = (\pi(f)\pi(g)\xi,\eta).$$

This shows the homomorphism property of $\pi$. To prove that $\pi$ is a $*$ homomorphism, we have to check that $\pi(\bar{f}) = \pi(f)^*$. This is equivalent to showing that if $f$ is a real-valued function, $\pi(f)^* = \pi(f)$ (check for yourself). We already know that $\hat{a} \in C(X)$ being real valued is equivalent to $a^* = a$. Hence we have in this case

$$\int \hat{a}\ d\mu_{\xi,\eta} = (a\xi,\eta) = \overline{(a\eta,\xi)}.$$

As any complex-valued measure is already completely determined by its values on real-valued continuous functions, by $\mathbf{C}$-linearity, we conclude that $\mu_{\eta,\xi} = \bar{\mu}_{\xi,\eta}$. Hence, if $f$ is a real-valued measurable function, we have

$$(\pi(f)\xi,\eta) = \int f\ d\mu_{\xi,\eta} = \overline{\int f\ d\mu_{\eta,\xi}} = \overline{(\pi(f)\eta,\xi)} = (\xi,\pi(f)\eta).$$

This concludes our proof.

**Definition** Let $X$ be a set with a Borel algebra $\Omega$, and let $\mathcal{H}$ be a Hilbert space. A *spectral measure* assigns to each element $U \subset \Omega$ a projection $E(U)$ such that

(a) $E(U_1)E(U_2) = E(U_1 \cap U_2)$, and $E(U_1)E(U_2) = 0$ if $U_1 \cap U_2 = \emptyset$,

(b) if $\{U_i, i \in \mathbf{N}\}$ are pairwise disjoint sets, then $E(\bigcup_i U_i) = \sum_i E(U_i)$, where the sum on the right converges in weak operator topology.

**Corollary** Any abelian $C^*$ algebra $A \subset B(\mathcal{H})$ defines a spectral measure via the map

$$E(U) = \pi(\chi_U),$$

where $\chi_U$ is the characteristic function of the measurable set $U$, and $\pi : B(X) \to B(\mathcal{H})$ is as in the theorem.

*Proof*. As $\pi$ is a $*$ homomorphism, we have $\pi(\chi_U)^2 = \pi(\chi_U^2) = \pi(\chi_U)$ and $\pi(\chi_U)^* = \pi(\chi_U)$, i.e. $E(U)$ is indeed a projection. Property (a) of the definition follows from $\chi_{U_1}\chi_{U_2} = \chi_{U_1\cap U_2}$. Property (b) follows from the $\sigma$-additivity of the measures $\mu_{\xi,\eta}$ as follows:

$$(E(\bigcup_i U_i)\xi,\eta) = \mu_{\xi,\eta}(\bigcup_i U_i) = \sum_i \mu_{\xi,\eta}(U_i) = \sum_i (E(U_i)\xi,\eta).$$

**Spectral Theorem** Let $a \in B(\mathcal{H})$ be a normal operator with spectrum $\sigma(a)$. Let $\pi : B(\sigma(a)) \to B(\mathcal{H})$ be the homomorphism of Theorem 1. Then there exists an associated spectral measure $E$ which satisfies

(a) $E(U) \neq 0$ for any open, non-empty subset $G \subset \sigma(a)$,

(b) the element $b \in B(\mathcal{H})$ commutes with $a$ and $a^*$ if and only if $b$ commutes with each spectral projection $E(U)$, $U \subset \sigma(a)$ measurable.

*Proof*. Assume there exists an open $U \subset \sigma(a)$ for which $E(U) = 0$. There exists an $x \in U$ and an $\varepsilon > 0$ such that the closed ball $B_\varepsilon := B_\varepsilon(x) \cap \sigma(a)$ of radius $\varepsilon$, centered at $x$ is contained in $U$. Let $\hat{b}$ be a nonzero continuous function in $C(\sigma(a))$ such that $supp(\hat{b}) \subset B_\varepsilon(x) \cap \sigma(a)$, and let $b \in C^*(a)$ be the element whose Gelfand transform is $\hat{b}$. Then we have

$$0 = bE(U) = \pi(\hat{b}\chi_U) = \pi(\hat{b}) = b,$$

a contradiction to $\hat{b} \neq 0$. This shows (a).

If $b$ commutes with $a$ and $a^*$, then it also commutes with any polynomial in $a$ and $a^*$. As multiplication is continuous and the polynomials in $a$ and $a^*$ are dense in $C^*(a)$, $b$ also commutes with every element $u \in C^*(a)$. Hence we have for any $\xi, \eta \in \mathcal{H}$ that

$$\int \hat{u}\ d\mu_{b\xi,\eta} = (ub\xi, \eta) = (u\xi, b^*\eta) = \int \hat{u}\ d\mu_{\xi,b^*\eta},$$

i.e. $\mu_{b\xi,\eta} = \mu_{\xi,b^*\eta}$. Hence we get for any $f \in B(X)$ that

$$(\pi(f)b\xi, \eta) = \int f\ d\mu_{b\xi,\eta} = \int f\ d\mu_{\xi,b^*\eta} = (\pi(f)\xi, b^*\eta) = (b\pi(f)\xi, \eta),$$

for all $\xi, \eta$. Hence $\pi(f)b = b\pi(f)$ for all $f \in B(X)$. This holds, in particular, if $f = \chi_U$ is the characteristic function of a measurable set $U$. This shows one direction of claim (b). The other direction is a consequence of the fact that $a$ can be approximated by finite linear combinations of projections $E(U_i)$. Indeed, it is well-known that the identity function $z \in \sigma(a) \mapsto z$ is approximates by finite linear combinations of characteristic functions $\chi_{U_i}$. Applying $\pi$, we get a finite linear combinations of $E(U_i)$'s which approximate $a$. Hence if there is a linear combination $c = \sum \alpha_i E(U_i)$ such that $\|a - \sum \alpha_i E(U_i)\| < \varepsilon$, then we have $cb = bc$ by assumption and hence

$$\|ba - ab\| \leq \|ba - bc + cb - ab\| \leq 2\|b\|\|c - a\| < 2\|b\|\varepsilon.$$

The claim follows from this.

**Remark** Often the element $\pi(f)$ in Theorem 1 is also denoted by $\int f\ dE$. This refers to the fact that one can define such an element for *any* spectral measure which may not necessarily come from an abelian $C^*$ algebra. Indeed, we can use finite linear combinations of the form $c = \sum_i \alpha_i \chi_{U_i}$ to approximate the Borel function $f$. Then, as outlined in the proof of the previous theorem for a special case, the corresponding operators $\pi(c) = \sum \alpha_i E(U_i)$ form a Cauchy sequence which converges to an operator $\int f\ dE$; see e.g. Prop IX.1.10 and 1.12 in the course book.

As an easy consequence of the spectral theorem, we can now show that while a spectral value $\lambda$ of a normal operator $a$ is usually not an eigenvalue of $a$, we can find vectors arbitrarily close to being an eigenvector with eigenvalue $\lambda$

**Corollary** Let $a \in B(\mathcal{H})$ be a normal operator. Then $\lambda \in \sigma(a)$ if and only if for every $\varepsilon > 0$ there exists $0 \neq \xi \in \mathcal{H}$ such that $\|a\xi - \lambda\xi\| \leq \varepsilon\|\xi\|$.

*Proof*. Let $\lambda \in \sigma(a)$, and let $U = U_\varepsilon(\lambda) \subset \mathbf{C}$ be the disk of radius $\varepsilon$ and center $\lambda$. If $\chi_U$ is the characteristic function of $U$, it follows that $|(z - \lambda)\chi_U(z)| \leq \varepsilon$ (just check for $z \in U$ and $z \notin U$ separately). By the spectral theorem, it follows that $E(U) = \pi(\chi_U)$ is a nonzero projection. But then we have for any $\xi \in E(U)\mathcal{H}$ that

$$\|(a - \lambda)\xi\| \ = \ \|\pi((z - \lambda)\chi_U)\xi\| \ \leq \ \|(z - \lambda)\chi_U\|_\infty\|\xi\| \ \leq \ \varepsilon\|\xi\|.$$

To prove the other direction, let $\lambda$ not be in $\sigma(a)$. Then we have for any vector $\xi$ that

$$\|\xi\| = \|(a - \lambda)^{-1}(a - \lambda)\xi\| \leq \|(a - \lambda)^{-1}\|\|(a - \lambda)\xi\|.$$

Hence there exists an $\varepsilon > 0$ such that $\|(a - \lambda)\xi\| > \varepsilon\|\xi\|$ for all nonzero $\xi \in \mathcal{H}$.

We can give a more concrete description of $\pi(B(X))$ in terms of $\pi(C(X)) = A$. For this recall the following result, which we proved earlier in the course (see also the book, Prop V.4.1):

**Proposition** Let $X$ be a normed vector space with unit ball $X_1$. Then the canonical embedding of $X_1$ into its second dual $X^{**}$ is dense its unit ball $X_1^{**}$ with respect to the weak $*$ topology on $X^{**}$.

**Theorem 3** The image $\pi(B(X))$ is contained in the closure of $A = \pi(C(X)) \subset B(\mathcal{H})$ in the weak operator topology.

*Proof*. Let $f \in B(X)$, and let $\mu$ be a regular Borel measure. Then $f$ induces a bounded linear functional on the space of regular Borel measures (which we can identify with $C(X)^*$) by $\mu \mapsto \int f\ d\mu$. Hence we can identify $f$ with an element in the second dual of $C(X)$. By the just mentioned proposition, there exists a bounded net $(a_i) \subset A$ such that $\int \hat{a}_i\ d\mu \to \int f\ d\mu$ for all measures $\mu$. Taking $\mu = \mu_{\xi,\eta}$ for two vectors $\xi, \eta \in \mathcal{H}$, it follows that

$$(a_i\xi, \eta) = \int \hat{a}_i\ d\mu_{\xi,\eta} \to \int f\ d\mu_{\xi,\eta} = (\pi(f)\xi, \eta).$$

Hence $\pi(B(X))$ is contained in the closure of $A$ in weak operator topology.

## VON NEUMANN ALGEBRAS

It can be shown that the image of $B(X)$ in Theorem 3 does coincide with the closure of $A$ in the weak operator topology. In any case, the last theorem shows that algebras which are closed in the weak operator topology contain lots of projections.

**Definition** A von Neumann algebra is a $C^*$ subalgebra $M \subset B(\mathcal{H})$ which is closed in the weak operator topology.

**Proposition** Let $S \subset B(\mathcal{H})$, and let $S' = \{a \in B(\mathcal{H}), as = sa \text{ for all } s \in S\}$. If $S$ is closed under the $*$ operation, i.e. $s \in S$ also implies $s^* \in S$, then $S'$ is a von Neumann algebra.

*Proof*. This is easy. We only give the proof that $S'$ is *wot* closed. So let $(a_i) \subset S'$ be a net converging to $a$ in weak operator topology, and let $s \in S$. Then we have, for $\xi, \eta \in \mathcal{H}$, that

$$(as\xi, \eta) = \lim_i (a_i s\xi, \eta) = \lim_i (sa_i\xi, \eta) = \lim_i (a_i\xi, s^*\eta) = (a\xi, s^*\eta).$$

Hence $(as\xi, \eta) = (sa\xi, \eta)$ for all $\xi, \eta \in \mathcal{H}$, which implies $a \in S'$.

The last proposition allows us to produce many examples. However, without additional information, it is often difficult to say much about their nature. As $S'$ is obviously closed under the $*$-operation, also the second commutant $S'' = (S')'$ is a von Neumann algebra. It follows directly from the definitions that $S \subset S''$. We will need the following lemma for the proof of von Neumann's bicommutant theorem. It is an exercise in matrix computation.

**Lemma** Let $A \subset B(\mathcal{H})$. Let $\pi$ be the obvious representation of $A$ on $\mathcal{H}^n$ given by $\pi(a)(\xi_i)_{i=1}^n = (a\xi_i)_{i=1}^n$. Then we have

(a) $\pi(A)' = \{(x_{ij}),\ x_{ij} \in A',\ 1 \le i, j \le n\}$,

(b) $\pi(A)'' = \{\pi(c),\ c \in A''\}$, where $\pi(c)(\xi_i)_{i=1}^n = (c\xi_i)_{i=1}^n$.

**Von Neumann's bicommutant theorem** Let $A \subset B(\mathcal{H})$ be a $*$-algebra. Then $A'' = \overline{A}^{sot}$. In particular, $\overline{A}^{sot}$ is a von Neumann algebra.

*Proof*. Let $c \in A''$. We have to show that for every $\varepsilon > 0$, $\xi_1,\ \xi_2,\ ..., \xi_n \in \mathcal{H}$ there exists $a \in A$ such that $\|(c - a)\xi_i\| < \varepsilon$, $1 \le i \le n$. Let us first do this in the special case $n = 1$, with $\xi = \xi_1$. Let $\mathcal{K} = \overline{A\xi}$, where $A\xi = \{a\xi,\ a \in A\}$, and let $p$ be the orthogonal projection onto $\mathcal{K}$. Then we can write any vector $\eta \in \mathcal{H}$ uniquely in the form

$$\eta = \eta_1 + \eta_2, \quad \eta_1 \in \mathcal{K}, \eta_2 \in \mathcal{K}^\perp. \qquad (*)$$

In view of this decomposition $p\eta = \eta_1$. As $A$ is a $*$ algebra with $A\mathcal{K} \subset \mathcal{K}$, we also have $A\mathcal{K}^\perp \subset \mathcal{K}^\perp$. Hence $a\eta = a\eta_1 + a\eta_2$ gives the decomposition $(*)$ for $a\eta$, for any $a \in A$, and we have

$$p(a\eta) = a\eta_1 = a(p\eta), \quad \eta \in \mathcal{H}.$$

Hence $p \in A'$. So if $c \in A''$, $cp = pc$ and hence $c\mathcal{K} = cp\mathcal{H} \subset p\mathcal{H} = \mathcal{K}$; in particular $c\xi \in \mathcal{K}$. As $A\xi$ is dense in $\mathcal{K}$ it follows that we can find for every $\varepsilon > 0$ an element $a \in A$ such that $\|c\xi - a\xi\| < \varepsilon$. This proves the claim for $n = 1$.

The general case $n > 1$ is reduced to this case by letting $A$ act on $\mathcal{H}^n$. Let $\xi = (\xi_1, \ ..., \ \xi_n)$, and let $c \in A''$. Then we show as in the previous paragraph that there exists $a \in A$ such that $\|\pi(c)\xi - \pi(a)\xi\| < \varepsilon$, i.e. we have

$$\sum_{i=1}^{n} \|c\xi_i - a\xi_i\|^2 \ < \ \varepsilon^2.$$

This proves that $\overline{A}^{sot}$ coincides with the von Neumann algebra $A''$.

**Definition** (a) For any algebra $A$, the center $\mathcal{Z}(A)$ consists of all elements $a \in A$ for which $za = az$ for all $a \in A$.

(b) We say that a von Neumann algebra $M$ is a factor if its center only consists of multiples of 1.

Let now $A = \bigotimes_{i=1}^{\infty} M_k$ be a $UHF$ $C^*$-algebra with unique normalized trace $tr$. Then $tr(a^*a) > 0$ for any $0 \neq a \in A$ (why?). Hence we have an injective embedding of $A$ into $L^2(A, tr)$, the Hilbert space completion of $A$ under the inner product $(a, b) = tr(b^*a)$. We denote the image of $a \in A$ in $L^2(A, tr)$ by $a_\xi$.

**Proposition** Let $A$ be a $UHF$ algebra with trace $tr$, acting on the Hilbert space $L^2(A, tr)$. Then we have

(a) The von Neumann algebra $A''$ is a factor with unique normalized *wot*-continuous trace $tr$ extending the trace on $A$.

(b) The set of values $tr(p)$, $p \in A''$ a projection, coincides with the unit interval $[0, 1]$.

*Proof*. With the notations above, and $a \in A$, we have

$$(a1_\xi, 1_\xi) = (a_\xi, 1_\xi) = tr(a). \qquad (*)$$

Hence we can extend the definition of $tr$ to any $a \in B(L^2(A, tr))$ via the left-hand side of $(*)$. Moreover, if a net $(a_i) \subset A$ converges to $a$ in *wot* topology, we have $tr(a_i) = tr(a)$. This also implies that $tr$ has the trace property on $\overline{A}^{wot} = A'' = \overline{A}^{sot}$.

Let now $z \in \mathbf{Z}(A'')$. Then, by the spectral theorem, also any spectral projection $p$ of $z$ is in $\mathbf{Z}(A'')$. Using this and the trace property, one easily checks that also $tr_p : a \in A \mapsto tr(pa)$ defines a trace on $A$. By uniqueness of the trace, $tr_p$ has to be a multiple of $tr$. As $tr_p(1) = tr(p) = tr(p)tr(1)$, this multiple is $tr(p)$. Hence we have

$$tr(pa) = tr(p)tr(a), \quad \text{for all } a \in A.$$

Let now $(a_i) \subset A$ be a net approximating $1 - p$ in *wot*. Then we have

$$0 = tr(p(1-p)) = \lim tr(pa_i) = \lim tr(p)tr(a_i) = tr(p)tr(1-p).$$

As $tr(e) = tr(e^*e) > 0$ for any projection $e \neq 0$, either $p$ or $1 - p$ is equal to 0. This implies that $z$ must be a multiple of 1.

Now let $(e_i)$ be a countable collection of projections with $e_ie_j = 0$ for $i \neq j$. Then also $\sum_{i=1}^{n} e_i$ is a projection with norm 1, and we have

$$(\xi, \xi) \geq (\sum_{i=1}^{n} e_i\xi, \xi) = \sum_{i=1}^{n} \|e_i\xi\|^2.$$

Hence the sum $\sum_i^\infty e_i\xi$ converges in norm for any vector $\xi$, i.e. $\sum_i^\infty e_i$ converges in *sot*.

To prove part (b), let $x \in [0,1]$ and let $x = \sum_i j_i/k^i$, $0 \leq j_i < k$ be its $k$-adic expansion (e.g if $k = 2$ it would be the binary expansion of $x$). We leave it to the reader to construct mutually orthogonal projections $e_i \in \bigotimes^\infty M_k$ such that $tr(e_i) = j_i/k$ (this can be done via tensor products of diagonal matrices where only the first $i$ factors are not equal to 1). This finishes the proof, as

$$tr(\sum e_i) = \sum tr(e_i) = \sum j_i/k^i = x.$$

Recall that the possible values of $tr(p)$, $p$ a projection in $\bigotimes^\infty M_k$ were given by the set $\{j/k^i, 0 \leq j \leq k^i, i \in \mathbf{N}\}$. This was used to show that we have many non-isomorphic $UHF$ algebras. Obviously, we can not use this method anymore to show that their closures in *wot* are isomorphic. It turns out that we could not do it by any other method either.

**Definition** (a) A von Neumann algebra $M$ is called *hyperfinite* if there exists a sequence of finite dimensional $C^*$ algebras $A_1 \subset A_2 \subset$ ... such that $M = \overline{\bigcup A_n}^{wot}$.

(b) A von Neumann factor is called of type $\mathrm{II}_1$ if it has a finite trace and is infinite dimensional. It is called of type $\mathrm{II}_\infty$ if it is of the form $M \otimes B(\mathcal{H})$, where $M$ is a $\mathrm{II}_1$ factor and $\mathcal{H}$ is an infinite dimensional Hilbert space.

It follows from the previous proposition that we have constructed a hyperfinite $\mathrm{II}_1$ factor. A factor of type I is isomorphic to $B(\mathcal{H})$ for $\mathcal{H}$ a Hilbert space. All other factors are called type III factors. One of the reasons why we were careful with showing the existence of the trace also on the weak closure of $A$ in the proof of the proposition is that this is not generally true. E.g if we take $A = \bigotimes^\infty M_2$ and $d$ is the diagonal $2 \times 2$ matrices with diagonal entries $1/(1+\lambda)$ and $\lambda/(1+\lambda)$, $0 < \lambda < 1$, we can perturb the usual trace to a so-called product state $\varphi_\lambda$ via

$$\varphi_\lambda(\bigotimes a_i) = \prod tr(da_i).$$

One can then show that the $GNS$ construction with respect to $\varphi_\lambda$ produces a so-called type $\mathrm{III}_\lambda$ factor. Observe that even though $A$ did have a trace, its weak closure for this particular $GNS$ construction does not have one anymore. Going back to our $\mathrm{II}_1$ factors, we have the following classical result, which shows that all the factors we constructed in the proposition are isomorphic:

**Theorem** (Murray-von Neumann) There exists only one hyperfinite $\mathrm{II}_1$ factor up to isomorphism.

Even though a von Neumann algebra is given as a concrete algebra acting on a Hilbert space, it is useful to consider actions of it on other Hilbert spaces. To compare these actions one defines a dimension function. If $M$ is a $\mathrm{II}_1$ factor, it naturally acts on $L^2(M, tr)$ via $GNS$-construction. As left multiplication commutes with right multiplication, we also obtain $M$-submodules $L^2(Mp, tr)$ of $L^2(M, tr)$. We now define the dimension function by

$$\dim_M L^2(Mp, tr) = tr(p), \quad p \in M \text{ a projection.}$$

If $M$ acts on an arbitrary Hilbert space $\mathcal{K}$, we always assume that $M$ acts unitally on a Hilbert space, i.e. $1 \in M$ acts as the identity on $\mathcal{K}$, and that the image of $M \subset B(\mathcal{K})$ is closed in *wot*. Then we say that the actions of $M$ onto Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic if there exists a bijective isometry $\Psi : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\Psi(a\xi) = a\Psi(\xi), \quad \xi \in \mathcal{H}_1,\ a \in M.$$

**Theorem** Let $M$ be a $\mathrm{II}_1$ factor acting on a Hilbert space $\mathcal{H}$. Then this action is either isomorphic to an infinite direct sum of copies of $L^2(M, tr)$, or there exists $n \in N$ and $p \in M$ such that

$$\mathcal{H} \cong L^2(Mp, tr) \oplus \bigoplus_{i=1}^{n} L^2(M, tr).$$

This can be used to get a well-defined dimension function $\dim_M(\mathcal{H})$ which is equal to $\infty$ in the first case, and $\dim_M \mathcal{H} = n + tr(p)$ in the second case.

A *subfactor* $N \subset M$ is an inclusion of a von Neumann factor $N$ into a bigger factor $M$ such that the 1 of $N$ coincides with the 1 of $M$. We will primarily be interested in $\mathrm{II}_1$ factors. One of the goals is to classify of all such inclusions up to automorphisms of $M$. If $M = M_k$ is a finite dimensional factor, it is not hard to show that $N$ must be isomorphic to $M_d$ with $d|k$, and there is only one such inclusion up to automorphisms. It can be helpful to keep this simple example in mind to get a feel for some of the following notions.

One of the first invariants of such inclusions is the *Jones index* given by

$$[M : N] = \dim_N L^2(M, tr);$$

we will in future just write $\dim_N M$ in this case.

**Simple Examples** (a) For the embedding $N = \mathbf{C} = \mathbf{C}1 \subset M_k = M$, the index $[M : N] = \dim_{\mathbf{C}} M_k = k^2$. Embedding $N = M_d$ via block diagonal matrices (with identical diagonal blocks) into $M = M_k$, one can similarly show that $[M : N] = (k/d)^2$.

(b) One gets a subfactor of the hyperfinite $\mathrm{II}_1$ factor $R = \overline{\bigotimes^\infty M_k}^{wot}$ from the obvious embeddings

$$M_d \otimes \bigotimes_{i=2}^{\infty} M_k \ \subset \bigotimes_{i=1}^{\infty} M_k.$$

Again, the index is $(k/d)^2$.

To get a more interesting example, we need the notion of *outer automorphism*. An automorphism $\alpha$ of the von Neumann algebra $M$ is called *inner* if there exists a unitary $u \in M$ such that $\alpha(a) = uau^*$ for all $a \in A$. It is called outer, if it is not inner. We say that we have an outer action of the group $G$ on $M$ if we have outer automorphisms $\alpha_g$ for each $g \in G$, $g \neq 1$ such that $\alpha_g \alpha_h = \alpha_{gh}$ and $\alpha_1 = id_M$.

**Theorem** Let $G$ be a finite group with an outer action on the factor $M$. Let $M^G = \{a \in M, \alpha_g(a) = a$ for all $g \in G\}$. Then we have $[M : M^G] = |G|$.

*Exercise* Let $R = \overline{\bigotimes^\infty M_2}^{wot}$, and let $d$ be the diagonal matrix with diagonal entries 1 and $\theta = e^{2\pi i/\ell}$. Show that the map

$$\alpha(\bigotimes_i a_i) \ = \ \bigotimes_i d a_i d^*$$

defines an outer automorphism on $R$, and that $i \in \mathbf{Z}/\ell \mapsto \alpha^i$ defines an outer action of $\mathbf{Z}/\ell$ on $R$.

Our examples so far all have integer index. However, there are many more examples. The following fundamental result gives a first glimpse of the rich structure related to subfactors:

**Theorem** (Jones) The possible values for an index $[M : N]$ of an inclusion $N \subset M$ of $\mathrm{II}_1$ factors is given by the set $\{4\cos^2 \pi/\ell,\ \ell = 3, 4, 5,\ \ldots\} \cup [4, \infty]$.

*Remarks* 1. The construction of interesting examples of subfactors is usually a highly nontrivial and involved endeavor. Jones' example for index $< 4$ were obtained as follows: Jones constructed a sequence of projections $e_1$, $e_2$, $e_3$ satisfying the relations $e_i e_j = e_j e_i$ if $|i - j| > 1$, and $e_i e_{i\pm1} e_i = \tau e_i$, where $\tau = 1/4\cos^2(\pi/\ell)$. It turns out that these projections generate an $AF$ algebra with a unique trace $tr$. E.g. if $\ell = 5$, $4\cos^2 \pi/5 = (3 + \sqrt{5})/2 = \lambda^2$, where $\lambda = (1 + \sqrt{5})/2$ is the golden ratio, and the $AF$ algebra is the one studied in a previous example. It was shown for this example that it has a unique trace, using the Perron-Frobenius

theorem; the same proof works for all other values of $\ell$. If $\pi$ is the representation of this $AF$ algebra with respect to this trace $tr$, we get an inclusion

$$N = \pi(\langle e_2, e_3, \ \dots \rangle)'' \subset M = \pi(\langle e_1, e_2, \ \dots \rangle)'' \quad \text{with } [M : N] = 4\cos^2 \pi/\ell.$$

2. The examples in Jones' paper for index $\geq 4$ have nontrivial relative commutants, i.e. $N' \cap M$ consists of more than scalar multiples of the identity. Subfactors of the hyperfinite $\mathrm{II}_1$ factor with trivial relative commutants have been classified up to index 5 in fairly recent work involving several researchers, including Jones, Morrison, Snyder, Peters and others. The classification up to index 4 has been known for a longer time and can be fairly easily stated:

**Theorem** The subfactors of the hyperfinite $\mathrm{II}_1$ factor of index less than 4 are classified by the Dynkin diagrams $A_n$, $n \geq 2$, $D_{2n}$, $n \geq 2$, $E_6$ and $E_8$. There is exactly one subfactor for each diagram, up to isomorphism, except for $E_6$ and $E_8$ where we have two nonisomorphic subfactors for each diagram.

*Remark* We briefly explain where these diagrams come from. Having an inclusion $N \subset M$, we can define relative tensor products

$$M^{\otimes n} = M \otimes_N M \otimes M \otimes_N \ \dots \ \otimes_N M \quad (n \text{ factors}),$$

where, as usual, $M \otimes_N M = M \otimes M/I$, where $I$ is the subspace of $M \otimes M$ generated by all elements of the form

$$m_1 n \otimes m_2 - m_1 \otimes n m_2, \quad m_1, m_2 \in M, \ n \in N.$$

It is easy to see that each $M^{\otimes n}$ is an $X - Y$ bimodules for any choice of $X, Y \in \{N, M\}$, acting via multiplication on the left and on the right. One can show that each $M^{\otimes n}$ can be decomposed into a direct sum of finitely many simple $X - Y$ bimodules for any fixed choice of $X$ and $Y$, provided that $[M : N] < \infty$. We will assume finite index in the following.

**Definition** A subfactor $N \subset M$ is called a finite-depth subfactor if, for every choice of $X$ and $Y$, there exists a finite set of irreducible $X - Y$ bimodules such that every irreducible $X - Y$ submodule of $M^{\otimes n}$ is isomorphic to one of these finitely many modules.

If we view an irreducible $M - N$ module $K$ as $N - N$ bimodule, it decomposes into a direct sum of finitely many irreducible $N - N$ bimodules. Using this, we can define the induction-restriction graph for such bimodules as follows: The vertices are labeled by equivalence classes of irreducible $M - N$ and $N - N$ bimodules. The vertex of a irreducible $M - N$ bimodule $K$ is connected with the vertex of the irreducible $N - N$ bimodules $L$ by $g$ edges, where $g$ is the number of irreducible $N - N$ bimodules isomorphic to $L$ in the decomposition of $K$ into a direct sum of irreducible $N - N$ bimodules. One can define a similar induction-restriction graph for $M - N$ and $M - M$ bimodules.

**Theorem** The just defined induction-restriction graphs define additional invariants for the inclusion $N \subset M$. For a subfactor of index $< 4$ these two graphs are the same, and those are the graphs mentioned in the previous theorem.