MATH 110
EXAM #1

Please answer the following questions. Because this test is open book and open note, you *will not get credit* for answers unless you demonstrate how you arrived at them. In short, please show all work.

Problem 1.

Consider the heat flow for an insulated rod:

$$\partial_t u = \partial_x^2 u \ ,$$
$$\partial_x u(t,0) \ = \ \partial_x u(t,\ell) \ = \ 0 \ ,$$
$$u(0,x) \ = \ f(x) \ .$$

The "total heat" in the rod at time $t = 0$ is defined to be:

$$H(0) \ = \ \int_0^\ell \ f(x) \ dx \ .$$

That is, just the integral of the initial heat distribution over the rod. Show that in this case the total heat distribution remains *constant* for all times $0 \leqslant t$. That is, if we define:

$$H(t) \ = \ \int_0^\ell \ u(t,x) \ dx \ ,$$

then we must have $H(t) = H(0)$. (Hint: This is similar to the energy estimates we derived in class. The goal in this case is to show that $\frac{d}{dt}H(t) = 0$.)

Problem 2.

Please answer the following:

a) Suppose that one has two solutions $u(x,t)$ and $v(x,t)$ to the heat equation with Dirichlet boundary conditions on the interval $[0,\pi]$:

$$u_t \ = \ u_{xx} \ ,$$
$$u(0,t) \ = \ u(\pi,t) \ = \ 0 \ ,$$

and:

$$v_t \ = \ v_{xx} \ ,$$
$$v(0,t) \ = \ v(\pi,t) \ = \ 0 \ .$$

Suppose that at the initial time $t = 0$ one has the inequality:

$$v(x,0) \ \leqslant \ u(x,0) \ .$$

Show that for *all* positive times $0 < t$ one has that:

$$v(x,t) \leqslant u(x,t) \ .$$

Please explain carefully your answer. (Hint: Consider the function $w = u - v$.)

b) Notice that the function:

$$u(x,t) = e^{-t}\sin(x) \ ,$$

solves the above heat equation with zero Dirichlet boundary conditions on the interval $[0,\pi]$. Show that if $v(x,t)$ is *any* solution to the (same) heat equation boundary value problem:

$$\begin{aligned} v_t &= v_{xx} \ , \\ v(0,t) &= v(\pi,t) = 0 \ , \end{aligned}$$

with the additional property that at $t = 0$ (on $[0,\pi]$):

$$-\sin(x) \leqslant v(x,t) \leqslant \sin(x) \ ,$$

then one always has:

$$|v(x,t)| \leqslant e^{-t} \ .$$

(Hint: Recall that if $-u \leqslant v \leqslant u$, with $0 \leqslant u$, then one also has $|v| \leqslant u$.)

Problem 5.

Calculate an explicit solution to the following Dirichlet problem in a unit square with continuous boundary values:

$$\begin{aligned} \Delta u &= 0 \ , && \text{in } \ 0 < x < 1 \ , \ 0 < y < 1 \ , \\ u(x,0) &= 0 \ , \\ u(1,y) &= y \ , \\ u(x,1) &= 1 + \sin(\pi x) \ , \\ u(0,y) &= y \ . \end{aligned}$$

(Hint: Decompose this problem into two: One where you remove $\sin(\pi x)$ from the boundary conditions, and a second one where the boundary values are zero except for $u(x,1) = \sin(\pi x)$. It is possible to guess the solution in the first case, while we know how to do the second problem. Get the solution of the original problem from those two solutions).

Problem 6.

a) Solve the following Dirichlet problem in the unit circle:

$$\begin{aligned} \Delta u &= 0 \ , && \text{in } \ 0 \leqslant r < 1 \ , \\ u(1,\theta) &= 1 + \sin(2\theta) + \cos(2\theta) \ . \end{aligned}$$

b) Verify that the maximum principle is true for this explicit solution. (Hint: It might be easiest to first subtract off a constant. Notice that if u is any function, then $u - C$ will have its maximum at the *same* point that u does, and that the maximum of $u - C$ is just the maximum of u minus C.)