MATH 110 PRACTICE FINAL

Please justify all your steps!

1. (a) Calculate the cosine series of the function x on the interval $[0, 1]$. You should do this using integration by parts.
   (b) Let A_n be the coefficients of the cosine series of x, as in (a). Calculate
   $$A_0^2/4 + \sum_{n=1}^{\infty} A_n^2.$$
   Hint: You should be able to do this even if you could not solve (a).
   (c) What is the value of the cosine series of (a) for $x = -1/2$? Again, you should be able to answer this question even if you could not solve (a).

2. (a) Find the solution of the heat equation $w_t = w_{xx}$ with initial and boundary conditions
   $$w(0,t) = w(\pi,t) = 0, \quad t > 0,$$
   $$w(x,0) = \sin(2x) + \sin(3x) \quad 0 < x < \pi.$$
   (b) Find the solution of the steady state problem $v_{xx} = 0$ with boundary conditions $v(0) = 0$ and $v(\pi) = 1$ (remember that for steady state temperature we have $v_t = 0$, i.e. we can consider v as a function of x only).
   (c) Find the solution of the heat equation $u_t = u_{xx}$ with initial and boundary conditions
   $$u(0,t) = 0 \quad u(\pi,t) = 1, \quad t > 0,$$
   $$u(x,0) = \sin(2x) + \sin(3x) + x/\pi, \quad 0 < x < \pi.$$

3. Let D be the disk $x^2 + y^2 \leq a^2$. We have shown in class that the eigenfunctions v of $-\Delta$ satisfying $-\Delta v = \lambda v$ are given by
   $$v(r,\theta) = J_m(\sqrt{\lambda} r)(A_m \cos m\theta + B_m \sin m\theta), \quad m = 0, 1, 2, ...$$

(a) What are the eigenfunctions $v(r,\theta) = f(r)$ which do not depend on θ and which satisfy the boundary condition $f'(a) = 0$? What are their eigenvalues? (It is OK to describe the eigenvalues as the zeros of one or several functions; it may not be possible to give a more explicit description).

(b) Let λ_n be the eigenvalues of (a) with eigenfunctions v_n. One can show that the function $f(r) = r^2$ can be written as
$$r^2 = \sum_{n=1}^{\infty} A_n v_n.$$
How does one calculate the coefficients A_n? It is OK to write down formulas for A_n involving integrals without calculating them.

More problems on the back side!

4. (a) Consider the two explicit functions:

$$\begin{aligned}\phi(x) &= 2\sin(\pi x) - 4\sin(2\pi x) + 3\sin(4\pi x) - 10\sin(5\pi x) + 5\sin(6\pi x),\\ \psi(x) &= -\sin(\pi x) + 6\sin(2\pi x) - 2\sin(5\pi x) + 3\sin(6\pi x) - \sin(7\pi x).\end{aligned}$$

Please compute the integral:

$$\langle \phi, \psi \rangle = \int_0^1 \phi(x)\psi(x)\ dx.$$

Hint : You may freely quote the values of certain integrals we have covered. The answer can be given quickly, but you need to explain clearly what you are doing. Do *not* use a calculator to get a numerical answer.)

(b) Let J_m be the m-th Bessel function of the first kind, and let $z_{m,n}$ be its n-th zero. Calculate

$$\int_0^a J_5(z_{5,3}r/a)J_5(z_{5,7}r/a)\ r\ dr.$$

Hint : Consider the theorem in Lecture 25 and its application to Bessel functions.

5. Consider the Laplace equation $\Delta u = 0$ on the square $[0,\pi] \times [0,\pi]$. We assume the boundary conditions

$$u_x(0,y) = 0 = u_x(\pi,y),$$

$$u(x,0) = 0 \quad \text{and} \quad u(x,\pi) = 3\cos 2x + 2\cos 5x.$$

(a) Calculate separated solutions $u(x,y) = X(x)Y(y)$ subject to the given homogeneous boundary conditions.
(b) Calculate the solution of the Laplace equation subject to all boundary conditions.

6. Consider the Laplace equation $\Delta u = 0$ on the upper semidisk D of radius 1, given in polar coordinates by $r \le 1$ and $0 \le \theta \le \pi$, subject to the boundary conditions $u(r,0) = 0 = u(r,\pi)$.
(a) Write down the ordinary differential equations for $R(r)$ and for $\Theta(\theta)$, with boundary conditions, for separated solutions $u(r,\theta) = R(r)\Theta(\theta)$.
(b) Calculate the separated solutions $u(r,\theta) = R(r)\Theta(\theta)$ in (a).