QUALIFYING EXAM: APPLIED ALGEBRA Spring 2007

The problems will count approximately the same. They will make up 60% of the whole exam. They are not necessarily ordered in degree of difficulty.

1. Let B be the matrix given by $B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 27 & 0 & 3 \end{bmatrix}$, and let $V = \mathbf{C}^3$ be an irreducible G-module for a finite group G, with $g \in G$ acting via the matrix A_g on V. Let $C = \frac{1}{|G|} \sum_g A_g B A_g^{-1}$.
   (a) Calculate $Tr(C)$, where Tr is the usual trace.
   (b) Calculate C.

2. Let G be a finite group, and let $p = \sum \alpha_g g$ be a minimal idempotent in the simple component of $\mathbf{C}G$ labeled by λ. Let d_λ be the dimension of a simple G-module on which p acts nonzero.
   (a) What is the dimension of the space $p\mathbf{C}S_n$? What is $\chi_{reg}(p)$, where χ_{reg} is the character of the left-regular representation.
   (b) Calculate the coefficient α_1.

3. (a) Expand the product of Schur functions $s_\lambda s_{[1^2]}$ into a linear combination of Schur functions for all Young diagrams λ with three boxes.
   (b) Decompose the simple S_5-module $V_{[3,2]}$ into a direct sum of simple $S_3 \times S_2$-modules, with S_3 permuting the letters $\{1, 2, 3\}$ and S_2 permuting the letters $\{4, 5\}$.
   (c) Determine the decomposition of $V_{[3,2]}$ as a direct sum of simple S_3-modules and determine the structure (i.e. decomposition into simple matrix algebras) of the commutant of the S_3 action on $V_{[3,2]}$ (where S_3 denotes the subgroup of S_5 leaving $\{4\}$ and $\{5\}$ fixed).

4. Let π be an $(n-1)$-cycle in S_n (e.g. (12) in S_3).
   (a) Calculate all S_n characters of an $(n-1)$-cycle for $n = 3, 4$.
   (b) Determine all Young diagrams λ with n boxes for which $\chi_\lambda(\pi) \neq 0$, for all $n \in \mathbf{N}$.

5. Consider the ideal I generated by the polynomials $x^3 - y^2 + 1, x^3 + y^2 + z^2 - 1$ and $y^2 + yz - 1$.
   (a) Calculate a Gröbner basis for the lexicographical order $x > y > z$.
   (b) Calculate all the solutions given by the common zeros of the generating polynomials, i.e. calculate the variety $V(I)$ of I.

6. Let $G = \mathbf{Z}/3$ act on a two dimensional vector space as a diagonal matrix with diagonal entries being $\theta^{\pm 1}$, where $\theta = e^{2\pi i/3}$.
   (a) Find a system of generators for $k[x, y]^G$.
   (b) Calculate the Hilbert series for $k[x, y]^G$.
   (c) Find at least one relation among the generators in (a). Give a precise description how you would find all possible relations (you need not carry out the calculations).