Math 120B **Test 2** **100 points** **May 22, 2013**

Directions: Justify all answers in detail. If you appeal to a theorem, show that the hypotheses of that theorem are justified. As usual, $z = x + iy$ and $w = f(z) = u + iv$. Each problem is worth 20 points.

(1) For $x > 2$, define $w = f(z) = \mathrm{Log}(iz/(z-2))$. Consider the interval $I = (2, \infty)$ on the real axis. Describe precisely the image $f(I)$ in the w-plane.

SOLUTION: For every positive s, the principle argument of is equals $\pi/2$. Thus $v = \pi/2$. Now, $u = \mathrm{Log}\,\dfrac{z}{z-2}$. When z is large, u is near $\mathrm{Log}\,1 = 0$, while when z is near 2, u is large. Thus $f(I) = \{u + i\pi/2 : u > 0\}$ in the w-plane.

(2) Define $w = f(z) = z^4$ in a neighborhood of $z_0 = 1 - i$. Define $g(w) = e^{(\log w)/4}$. In order that g be a conformal inverse function of f in a neighborhood of $w_0 = -4$, which branch of $\log w$ above would you take? (You should specify your branch by giving a range for $\arg w$).

SOLUTION: We have $1 - i = g(-4)$, so

$$\frac{1-i}{\sqrt{2}} = e^{i(\arg -4)/4}.$$

This holds when $\arg(-4) = -\pi$, so one can choose the branch of $\log w$ for which $-2\pi < \arg w < 0$.

(3) Let A and B be two smooth contours in the z-plane that intersect at the point $z = 0$, where the angle from A to B is 50 degrees. Let $w = f(z) = \cos z$. In the w-plane, what is the angle from the curve $f(A)$ to the curve $f(B)$ at the intersection point $w = 1$?

SOLUTION: Here f is entire, and its Taylor expansion about 0 begins $1 - z^2/2 + \ldots$, so 2 is the smallest positive value of m for which the m-th derivative of f is nonzero at the origin. (In particular, f fails to be conformal at the origin.) By problem 10 on page 363, the desired angle is $50m$, i.e., 100 degrees.

(4) A region of the z-plane (representing a thin plate) has steady state temperature $T(x, y) = 3yx^2 - y^3$. Find equations that describe the "lines of heat flow".

SOLUTION: Using the integral formula (5) on page 364, for example, one sees that a harmonic conjugate of $T(x, y)$ is $-x^3 + 3xy^2$. Thus the lines of flow are described by level curves of the form $-x^3 + 3xy^2 = constant$.

(5) Let $w = f(z) = u + iv$ be a conformal map with $f(0) = 0$. Suppose that the curve $u(x, y) = 0$ in the z-plane has a vertical tangent line at the point $z = 0$. Using the chain rule and Cauchy-Riemann equations, prove that the curve $v(x, y) = 0$ has a horizontal tangent line at the point $z = 0$.

SOLUTION: Suppose for the purpose of contradiction that $u_y \neq 0$ at the origin. Then the equation $u(x, y) = 0$ can be solved for y as a function of x in a neighborhood of the origin, by the Implicit Function Theorem. Differentiating the above equation with respect to x, we see that

$$\frac{dy}{dx} = \frac{-u_x}{u_y}$$

at the origin, which contradicts the "vertical tangent" hypothesis. Thus $u_y = 0$ at the origin, so by C-R, $v_x = 0$ at the origin. Since f is conformal, $v_y \neq 0$ at the origin. Then the equation $v(x, y) = 0$ can be solved for y as a function of x. Differentiating the above equation with respect to x, we see that

$$\frac{dy}{dx} = \frac{-v_x}{v_y}$$

at the origin. Since $v_x = 0$ at the origin, the tangent line in question is horizontal.