Math 120B Final 160 points June 13, 2013

Directions: Justify all answers. If you appeal to a theorem, show that the hypotheses of that theorem are justified. The closed unit disk is denoted by $D = \{z : |z| \leq 1\}$.

(1) (A) Let f be analytic on the disk $D = \{z : |z| \leq 1\}$. Using the Cauchy Integral Formula, prove that if $|f(0)|$ is the maximum value of $|f(z)|$ on D, then $|f(z)|$ is constant on D. (Extra Credit for showing the stronger conclusion that $f(z)$ is constant on D.) [15 pts]

SOLUTION: By (2) on page 175 with $z_0 = 0$ and $\rho \leq 1$,

$$|f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})| d\theta.$$

Thus

$$0 \leq \frac{1}{2\pi} \int_0^{2\pi} \{|f(\rho e^{i\theta})| - |f(0)|\} d\theta.$$

By hypothesis, the expression in braces is a nonpositive continuous function of θ, so it must equal 0 (otherwise the integral would be negative). Thus $|f(z)| = |f(0)|$ for every $z \in D$. For the extra credit question, see Example 4, p. 76.

(B) Let $u(z)$ be a harmonic function on D. Prove that if $u(0)$ is the maximum value of $u(z)$ on D, then $u(z)$ is constant on D. [10 pts]

SOLUTION: On a simply connected domain containing D, there is an analytic function $h(z) = u(z) + iv(z)$ for some harmonic function $v(z)$. Let $f = e^h$. Then f is analytic, and by hypothesis, the function $|f(z)| = e^{u(z)}$ on D has its max at $z = 0$. Thus $|f(z)|$ is constant on D by part (A), so $u(z)$ is constant on D.

(2) A very thin coin of radius 1 is represented by D. At each point $e^{i\phi}$ on the boundary of D, the temperature is $\cos(2\phi)$.

(A) Show that the steady state temperature at each point $x + iy \in D$ is given by $T(x, y) = x^2 - y^2$. [20 pts]

SOLUTION: Multiply both sides of (1) by $\frac{\cos(2\phi)}{2\pi}$ and integrate from $\phi = 0$ to $\phi = 2\pi$ to get the temperature at the point $re^{i\theta}$. The n-th summand has absolute value bounded by r^n, so the series converges uniformly by the Weierstrass M-test. Thus we can integrate term by term. Simplify $\cos(n\phi - n\theta)$ using (2), and then compute the integrals using (3), (4), (5). The only term that survives is for $n = 2$, which by (5) equals $r^2 \cos(2\theta) = x^2 - y^2$.

(B) Is the steady state temperature function $T(x, y)$ given in part (A) unique? Explain in detail. [15 pts]

SOLUTION: Yes, because if there were TWO continuous functions on D which were harmonic in the interior of D with the same boundary values, then their difference would be a continuous function $w(z)$ on D, harmonic in the interior of D, which vanishes on the boundary. By the maximum principle, neither w nor $-w$ can have a max in the interior of D. Thus their max occurs on the boundary, so w is identically zero. This proves uniqueness.

(C) Give the equations for the lines of heat flow. [15 pts]

SOLUTION: The analytic function z^2 equals $x^2 - y^2 + i(2xy)$. Thus the lines of heat flow are the level curves $2xy = k$, where k is constant. These are hyperbolas.

For problem (2), you may use each of these 5 facts without proof:

$$P(1, r, \phi - \theta) = 1 + 2\sum_{n=1}^{\infty} r^n \cos(n\phi - n\theta), \tag{1}$$

where P is the Poisson kernel and $r < 1$;

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b); \tag{2}$$

$$\int_0^{2\pi} \cos(m\phi)\sin(n\phi)d\phi = 0, \quad \text{for all integers } m \text{ and } n; \tag{3}$$

$$\int_0^{2\pi} \cos(m\phi)\cos(n\phi)d\phi = 0, \tag{4}$$

whenever n and m are DIFFERENT positive integers;

$$\int_0^{2\pi} \cos(n\phi)\cos(n\phi)d\phi = \pi, \quad \text{for every integer } n \geq 1. \tag{5}$$

(3) Let f denote an analytic function on the disk D which is nonzero at every point of D except at the center, $f(0) = 0$. Using Rouché's theorem, show in detail that there is an open disk centered at 0 which is completely contained in $f(D)$. [20 pts]

SOLUTION: Since $|f|$ is a continuous nonzero function on the boundary of D, $|f|$ has a minimum value $e > 0$ at some point on the boundary. We claim that the desired open disk E is the disk of radius e centered at 0. Fix $w \in E$ and let $g(z)$ be the constant function $-w$. Then for z on the boundary of D, $|f(z)| \geq e > |-w| = |g(z)|$. Thus by Rouché's theorem, $f(z) - w$ has a zero z in the interior of D, since f does. Thus every $w \in E$ is contained in $f(D)$.

(4) For z on the semicircle $I = \{e^{i\theta} : 0 < \theta < \pi\}$, define $w = f(z) = \text{Log}\{(1+z)/(1-z)\}$. Describe precisely the image $f(I)$ in the w-plane. *Hint*: Rationalize the denominator. [15 pts]

SOLUTION: When $z = e^{i\theta}$, the linear fractional transformation equals $\dfrac{i(1+\cos(\theta))}{\sin(\theta)}$. As θ varies, this traces out the positive imaginary axis. Applying the principal logarithm, we see that $f(I)$ is the horizontal line through the point $i\pi/2$.

(5) Define $w = f(z) = z^4$ in a neighborhood of $z_0 = i - 1$. Define $g(w) = e^{(\log w)/4}$. In order that g be a conformal inverse function of f in a neighborhood of $w_0 = -4$, which branch of $\log w$ above would you take? (You should specify your branch by giving a range for $\arg w$). [15 pts]

SOLUTION: We have $\frac{i-1}{\sqrt{2}} = e^{i\arg(-4)/4}$. The left side equals $e^{i3\pi/4}$. Thus we can take $\arg(-4)$ to be 3π. So a desired range is $2\pi < \arg w < 4\pi$.

(6) Use the residue theorem to show that

$$\int_0^\infty \frac{x^{1/3}dx}{(x+1)^2} = \frac{2\pi\sqrt{3}}{9}. \qquad [20 \text{ pts}]$$

SOLUTION: Use the contour on page 284. Let $\zeta = e^{2i\pi/3} = (-1+i\sqrt{3})/2$. The integrals on the large and small circles can be shown to approach 0, as shown on page 285. Let I denote the integral in problem (6). Then

$$(1-\zeta)I = 2i\pi Res_{z=-1}\frac{z^{1/3}}{(z+1)^2} = 2i\pi\frac{(e^{i\pi})^{-2/3}}{3} = \frac{2i\pi\bar{\zeta}}{3}.$$

Simplifying, we get the desired value of I.

(7) Let $f(z) = 6e^z + z - 1$. How many $z \in D$ satisfy $f(z) = 0$? [15 pts]

SOLUTION: Consider the circle $C = \{z : |z| = 1 + \epsilon\}$, where ϵ is a small positive number. For $z = x + iy \in C$,

$$|6e^z| = 6e^x \geq 6e^{-1-\epsilon} > 2 + \epsilon = |z| + 1 \geq |z-1|,$$

if ϵ is sufficiently tiny. Thus by Rouché's theorem, $6e^z$ and $6e^z + z - 1$ have the same number of zeros inside the circle C. But $6e^z$ never vanishes, so $6e^z + z - 1$ has no zeros at all in D.