NOTES

1. Oscillatory integrals - one dimensional theory

A fundamental object in Fourier analysis is the oscillatory integral

\[ I(\lambda) = \int_a^b e^{i\lambda \phi(x)} \psi(x) dx, \]

where \( \phi \) is a real-valued smooth function and \( \psi \) has compact support in \((a, b)\). The basic facts about can be presented in terms of three principles: localization, scaling and asymptotic.

1.1. Localization. This refers to the compact support property of \( \psi \) in \((a, b)\). The asymptotic behavior of \( I(\lambda) \) is determined by the critical points of \( \phi \), i.e. where \( \phi'(x) = 0 \). The main result of this section is the following:

**Proposition 1.** Let \( \phi \) and \( \psi \) be smooth functions so that \( \psi \) has compact support in \((a, b)\) and \( \phi'(x) \neq 0 \) for all \( x \in [a, b] \). Then

\[ |I(\lambda)| \lesssim N^{-N}, \]

for all \( N \geq 0 \).

Obviously the interesting information comes for \( \lambda \to \infty \). The proof goes as follows. We define

\[ Df(x) = (i\lambda \phi'(x))^{-1} \frac{df}{dx} \]

and let \( \tilde{D} \) be its transpose,

\[ \tilde{D}f(x) = -\frac{d}{dx} \left( \frac{f}{i\lambda \phi'(x)} \right). \]

We have \( D^N(e^{i\lambda \phi}) = e^{i\lambda \phi}, \forall N \), thus integrating by parts

\[ \int_a^b e^{i\lambda \phi(x)} \psi(x) dx = \int_a^b D^N(e^{i\lambda \phi(x)}) \psi(x) dx = \int_a^b e^{i\lambda \phi(x)} (\tilde{D})^N \psi(x) dx \]

from which the conclusion follows. Here we used that \( |\phi'| \geq c > 0 \) on \((a, b)\).

It is important to note that without the localization property of \( \psi \) the above estimate fails. Precisely let \( \phi(x) = x \) and \( \psi = 1 \) on \((a, b)\); then

\[ I(\lambda) = \int_a^b e^{i\lambda x} dx = \frac{e^{i\lambda b} - e^{i\lambda a}}{i\lambda} \]

which says that \( |I(\lambda)| \approx \lambda^{-1} \) for \( \lambda \) large.

However if we deal with

\[ \int_a^b e^{i\lambda \phi(x)} \psi(x) dx \]
and \( \phi, \psi \) have periodic boundary conditions, i.e. \( \phi^k(a) = \phi^k(b) \) and \( \psi^k(a) = \psi^k(b) \) for all \( k \) than we obtain decay of all orders since the boundary terms cancel each other.

1.2. Scaling. Now we would like to understand the behavior of \( I(\lambda) \) in the case \( \psi = 1 \),

\[
I(\lambda) = \int_a^b e^{i\lambda \phi(x)} dx
\]

and when \( \phi \) has critical points in \((a, b)\). We work under the hypothesis that \( |\phi^{(k)}(x)| \geq 1 \) for some fixed \( k \). We would like to obtain an estimate which is independent of \( a \) and \( b \). A simple scaling argument hints at what should be expected. If \( x \rightarrow \frac{1}{\lambda} x \) shows that the only possibility is a decay of type \( \lambda^{-\frac{1}{2}} \). This is indeed the case,

**Proposition 2.** Suppose \( \phi \) is real valued and smooth in \((a, b)\) and \( |\phi^k| \geq 1 \) in \((a, b)\) Then

\[
|I(\lambda)| \leq c_k \lambda^{-\frac{1}{2}}
\]

holds when:

i) \( k \geq 2 \) or

ii) \( k = 1 \) and \( \phi' \) is monotonic.

The constant \( c_k \) depends on \( k \) and \( \phi \).

**Proof.** By induction. Let \( k = 1 \). In that case we have

\[
I(\lambda) = \int_a^b D(e^{i\lambda \phi(x)}) dx = \int_a^b e^{i\lambda \phi(x)} \tilde{D}(1) dx + (i\lambda \phi'(x))^{-1} e^{i\lambda \phi(x)} |_a^b.
\]

The boundary terms come with the correct estimate, while for the remaining integral,

\[
| \int_a^b e^{i\lambda \phi(x)} (i\lambda)^{-1} \frac{d}{dx} \left( \frac{1}{\phi'} \right) dx | \leq \lambda^{-1} \int_a^b \left| \frac{d}{dx} \left( \frac{1}{\phi'} \right) \right| dx
\]

\[
= \lambda^{-1} \left| \int_a^b \frac{d}{dx} \left( \frac{1}{\phi'} \right) dx \right| = \lambda^{-1} \left| \frac{1}{\phi'(b)} - \frac{1}{\phi'(a)} \right| \lesssim \lambda^{-1}.
\]

where, in passing to the second line, we have used the monotonicity of \( \phi' \).

Next we assume that the result holds true for \( k \) and we prove it for \( k + 1 \). Without restricting the generality of the argument we can assume that \( \phi^{k+1}(x) \geq 1 \) on \([a, b]\). Therefore \( \phi^k \) is strictly increasing. Let \( c \in [a, b] \) be the unique point where \( |\phi^k(x)| \) assume its minimum value in \([a, b]\). If \( \phi^k(c) = 0 \) then \( |\phi^k(x)| \geq \delta \) for \( x \) outside \((c - \delta, c + \delta)\) (if \( \phi^k(c) \neq 0 \), then \( c = a \) or \( c = b \) and a similar argument will work). We write

\[
\int_a^b = \int_a^c - \int_{c-\delta}^{c+\delta} + \int_c^b
\]
Using the induction hypothesis,
\[ | \int_a^{c-\delta} e^{i\lambda \phi} dx | + | \int_{c-\delta}^b e^{i\lambda \phi} dx | \leq 2c_k (\lambda \delta)^{-\frac{1}{k}}. \]

By trivial estimation,
\[ | \int_{c-\delta}^{c+\delta} e^{i\lambda \phi} dx | \leq 2\delta \]

hence
\[ |I(\lambda)| \leq \frac{2c_k}{(\lambda \delta)^\frac{1}{k}} + 2\delta. \]

Optimizing with respect to \( \delta \) gives \( \delta = \lambda^{-\frac{1}{k+1}} \) and the conclusion follows. \( \square \)

A direct consequence is the following

**Corollary 1.** Under the assumptions on \( \phi \) as above and \( \psi \) having compact support in \((a, b)\) we have
\[ | \int_a^b e^{i\lambda \phi(x)} \psi(x) dx | \leq c_k \lambda^{-\frac{1}{k}} \left( |\psi(b)| + \int_a^b |\psi'(x)| dx \right). \]

Indeed we write
\[ \int_a^b e^{i\lambda \phi(x)} \psi(x) dx = \int_a^b F'(x) \psi(x) dx \]
where
\[ F(x) = \int_a^x e^{i\lambda \phi(x)} dx \]
and using the estimate \( |F(x)| \leq c_k \lambda^{-\frac{1}{k}}. \)

1.3. **Asymptotics.** In this section we fully describe the asymptotic behavior of
\[ I(\lambda) = \int_a^b e^{i\lambda \phi(x)} \psi(x) dx. \]

This depends on the critical points of \( \phi \), i.e where \( \phi'(x) = 0 \). Assume that the support of \( \psi \) is so small so that it contains only on critical point of \( \phi \) which we call \( x_0 \). The expansion now depends on the smallest \( k \geq 2 \) such that
\[ \phi^k(x_0) \neq 0. \]

We have the following result.

**Proposition 3.** Suppose \( k \geq 2 \) and \( \psi^k(x_0) \neq 0 \), while \( \psi(x_0) = \psi'(x_0) = \ldots = \psi^{(k-1)}(x_0) = 0. \) If \( \psi \) is supported in a sufficiently small neighborhood of \( x_0 \) then
\[ I(\lambda) \approx \lambda^{-\frac{1}{k}} \sum_{j=0}^\infty a_j \lambda^{-\frac{j}{k}}. \]
in the sense that
\[ \left| \frac{d}{d\lambda} \alpha [I(\lambda) - \lambda^{-\frac{1}{k}} \sum_{j=0}^{N} a_j \lambda^{-\frac{j}{k}}] \right| \lesssim \lambda^{-\alpha - \frac{N+1}{k}} \]

**Proof.** We split the argument in a few steps.

Step 1. We claim
\[
\int_{-\infty}^{\infty} e^{i\lambda x^2} x^l e^{-x^2} dx \approx \lambda^{\frac{l+1}{2}} \sum_{j=0}^{\infty} c_j \lambda^{-j}
\]
for any non-negative integer \( l \). Note that if \( l \) is odd, then the integral equals 0.

We rewrite the integral as
\[
\int_{-\infty}^{\infty} e^{-(1-i\lambda)x^2} dx = (1-i\lambda)^{-\frac{1}{2}-\frac{l}{2}} \int_{\gamma} e^{-z^2} z^l dz
\]
where in \( z = (1-\lambda i)^{\frac{1}{2}} x \) we use the principal branch of the logarithm and \( \gamma = (1-\lambda i)^{\frac{1}{2}} \cdot \mathbb{R} \). Using the fast decay of \( e^{-z^2} \) we can switch the contour of integration to \( \mathbb{R} \), hence we obtain
\[
(1-i\lambda)^{-\frac{1}{2}-\frac{l}{2}} \int_{-\infty}^{\infty} e^{-x^2} x^l dx.
\]

Then we write
\[
(1-i\lambda)^{-\frac{1}{2}-\frac{l}{2}} = \lambda^{-\frac{l+1}{2}} (\lambda^{-1} - i)^{-\frac{l+1}{2}}
\]
for \( \lambda > 0 \) and with \( w = \lambda^{-1} \) we use the power series expansion for \((w-i)^{-\frac{l+1}{2}}\) inside the disc \( |w| < 1 \).

Step 2. If \( \eta \in C_0^\infty \) then
\[
\left| \int_{-\infty}^{\infty} e^{i\lambda x^2} x^l \eta(x) dx \right| \lesssim \lambda^{-\frac{1}{2}-\frac{l}{2}}.
\]
Let \( \zeta \) be a \( C_0^\infty \) function with \( \zeta(x) = 1 \) for \( |x| \leq 1 \) and \( \zeta(x) = 0 \) for \( |x| \geq 2 \). We write
\[
\int_{-\infty}^{\infty} e^{i\lambda x^2} x^l \eta(x) dx = \int_{-\infty}^{\infty} e^{i\lambda x^2} x^l \eta(x) \zeta(x) dx + \int_{-\infty}^{\infty} e^{i\lambda x^2} x^l \eta(x) (1 - \zeta(x)) dx
\]
The first integral is trivially bounded by \( C \varepsilon^{l+1} \). For the second integral we use integration by parts and write it as
\[
\int e^{i\lambda x^2} (\bar{D})^N (x^l \eta(x) (1 - \zeta(x))) dx
\]
where \( \bar{D} f = -(i\lambda)^{-1} (f \frac{d}{dx})' \). From this we obtain that the second integral is bounded by
\[
\frac{C_N}{\lambda^N} \int_{|x| \geq \varepsilon} |x|^{l-2N} dx = C_N' \lambda^{-N} \varepsilon^{l-2N+1}.
\]
if $l - 2N < -1$. Hence we obtain a total bound of

$$C_N(\varepsilon^{l+1} + \lambda^{-N} \varepsilon^{l-2N+1})$$

which for $\varepsilon = \lambda^{-\frac{1}{2}}$ leads to the claim we made for $N > \frac{l+1}{2}$.

A similar result shows that

$$|\int e^{i\lambda x^2} g(x) dx| \lesssim \lambda^{-N}$$

for every $N > 0$ provided that $g \in S$ is supported away from the origin.

Step 3. We now prove the main result for $\phi(x) = x^2$. We write

$$\int_{-\infty}^{\infty} e^{i\lambda x^2} \psi(x) dx = \int_{-\infty}^{\infty} e^{i\lambda x^2} e^{-x^2} [e^{x^2} \psi(x)] \tilde{\psi}(x) dx$$

for some $\tilde{\psi} \in C_0^\infty$ which equals 1 in the support of $\psi$. For each $N$ we write the taylor series:

$$e^{x^2} \psi(x) = \sum_{j=1}^{N} b_j x^j + x^{N+1} R_N(x) = P(x) + x^{N+1} R_N(x)$$

which we then substitute into the original integrals to obtain three terms

$$\sum_j b_j \int e^{i\lambda x^2} e^{-x^2} x^j dx$$

$$\int e^{i\lambda x^2} x^{N+1} R_N(x) e^{-x^2} \tilde{\psi}(x) dx$$

$$\int e^{i\lambda x^2} P(x) e^{-x^2} (\tilde{\psi}(x) - 1) dx$$

For the first we use the result in Step 1, for the second we use the result in Step 2, while for the third we use the result mentioned at the end of Step 2.

Step 4. We finish the case $k = 2$. We write

$$\phi(x) = c(x - x_0)^2(1 + \varepsilon(x))$$

where $\varepsilon$ is smooth and $|\varepsilon(x)| \lesssim |x - x_0|$ for $x$ near $x_0$; in particular $|\varepsilon(x)| < \frac{1}{2}$ in a a small neighborhood of $x_0$. Also, $\phi'(x) \neq 0$ for $x \neq x_0$ and in a small neighborhood of $x_0$. Then within such a neighborhood $U$ we introduce the change of variables

$$y = (x - x_0)(1 + \varepsilon(x))^\frac{1}{2}.$$ 

This is a diffeormorphism from $U$ to a neighborhood of 0 and $cy^2 = \phi(x)$, and the integral above becomes

$$\int_a^b e^{i\lambda cy^2} \tilde{\psi}(y) dy$$

where $\tilde{\psi} \in C_0^\infty$ if the support of $\psi$ lies in $U$. Then the result follows from the previous arguments. □
From the above argument it follows that each constant in the asymptotic expansion depends on finitely many derivatives of the functions involved (at $x_0$). We now explain how to find $a_0$. Going back to step 4, we see that the change of variables is given by $y = c(x - x_0)(1 + \epsilon(x))$ with $c = \frac{\phi''(x_0)}{2}$. Based on this we introduced the new function $\tilde{\psi}$ with $\tilde{\psi}(0) = \psi(x_0)$. Next, at step 3 we use Taylor series of $e^{x^2 \tilde{\psi}(x)}$, from which we need only the first term and this is $\psi(x_0)$. Thus the first term in the asymptotic series is given by

$$( -i )^{-\frac{\lambda}{2}} \pi^{\frac{1}{2}} (c \lambda)^{-\frac{1}{2}} \psi(x_0) = ( \frac{2\pi}{-i \phi''(x_0)} )^{\frac{1}{2}} \psi(x_0) \lambda^{-\frac{1}{2}}.$$

A classical application of the above result lies in the asymptotic expansion for the Bessel functions:

$$J_m(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{ir \sin \theta} e^{-im \theta} d\theta$$

Here we take $\lambda = r$ and $\phi(x) = \sin x$. $\phi$ has two critical points at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ and at both these points $\phi'' = \pm 1$. We construct a smooth partition of unity $1 = \psi_1 + \psi_2 + \psi_3$ where $\psi_1$ has support near $\frac{\pi}{2}$ and $\psi_2$ near $\frac{3\pi}{2}$ and such that $|\phi''| \geq \frac{1}{2}$ in the support of $\psi_1$ and $\psi_2$. Using this partition we break $J_m$ into three integrals: for the first two we obtain decay of order $r^{-\frac{1}{2}}$ while for the third one of order $r^{-N}$ (here we use the periodicity of the functions involved), hence

$$|J_m| \lesssim r^{-\frac{1}{2}}.$$ 

However we have the more precise result for the first two terms and we use the computations above for the $a_0$ terms to obtain:

$$\frac{1}{2\pi} \left[ \left( \frac{2\pi}{i} \right)^{\frac{1}{2}} e^{-im \frac{\pi}{2}} e^{ir} + \left( \frac{2\pi}{-i} \right)^{\frac{1}{2}} e^{-im \frac{3\pi}{2}} e^{-ir} \right] r^{-\frac{1}{2}} + O(r^{-\frac{3}{2}})$$

$$= \left( \frac{2\pi}{\pi} \right)^{\frac{1}{2}} r^{-\frac{1}{2}} \cos \left( r - \frac{m\pi}{2} - \frac{\pi}{4} \right) + O(r^{-\frac{3}{2}}).$$

2. Oscillatory integrals - several variables theory

We want to develop a similar theory for the same object

$$I(\lambda) = \int e^{i\lambda \phi(x)} \psi(x) dx,$$

in higher dimensions. Unfortunately the theory is not so easy as in the 1D case. A critical point for $\phi$ is a point $x_0$ where $\nabla \phi(x_0) = 0$. In the absence of such points, we have a similar result to the one in 1D.

**Proposition 4.** Suppose $\psi$ is smooth, has compact support, and $\phi$ is a smooth real-valued function with no critical points in the support of $\psi$. Then

$$|I(\lambda)| \lesssim_N \lambda^{-N}.$$
for all $N \geq 0$.

**Proof.** We can split the domain of integration into finitely many pieces where $\xi \cdot \nabla \psi(x) \geq c > 0$. On each such piece we choose a new system of coordinates such that $x_1$ lies in the direction of $\xi$ and write

$$
\int e^{i\lambda \phi(x)} \psi(x) dx = \int \left( \int e^{i\lambda \phi(x_1, \ldots, x_n)} \psi(x_1, \ldots, x_n) dx_1 \right) dx_2 \ldots dx_n.
$$

For the inner integral we apply the result in Proposition 1 and the result follows. \hfill \square

The equivalent of Proposition 2 is the following

**Proposition 5.** Suppose $\psi$ is smooth and compactly supported, and $\phi$ is such that there is a multi-index $\alpha$ with $|\alpha| = k > 0$ such that $|\partial^\alpha \phi| \geq 1$ in the support of $\phi$. Then

$$
|I(\lambda)| \lesssim \lambda^{-\frac{1}{2}} (\|\psi\|_{L^\infty} + \|\nabla \psi\|_{L^1}).
$$

The idea of the proof is to break the domain in pieces where $|\xi \cdot \nabla^k \psi| \geq c > 0$ and then use the 1D argument on those pieces as above. This requires the following algebraic fact. Let $S$ be the space of homogeneous polynomials of degree $k$, for which a basis is given by $\{x^\alpha : |\alpha| = k\}$. Let $d(k, n)$ be its dimension. Then there are unit vectors $(\xi^j)_{j=1}^{d(k, n)}$ such that the homogeneous polynomials $(\xi^j \cdot x)^k$, $j = 1, \ldots, d(k, n)$ form also a basis.

The result from above is not optimal and this has to do with the technique we are using. One such example comes from the function $\phi(x) = x_1 \cdot x_2$, where the above results gives decay of order $\lambda^{-\frac{1}{2}}$, but the result below gives decay of order $\lambda^{-1}$.

We now discuss the case of non-degenerate critical points. Suppose $\phi$ has a critical point at $x_0$. If the symmetric matrix (the Hessian of $\phi$)

$$
\left[ \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right]_{i,j=1,n}(x_0)
$$

is invertible, the the critical point $x_0$ is said to be non-degenerate. Moreover, from the Taylor expansion it follows that the critical point is an isolated critical point. We have the following result

**Proposition 6.** Suppose $\phi(x_0) = 0$ and $\phi$ has a non-degenerate critical point at $x_0$. If $\psi$ is supported in a sufficiently small neighborhood of $x_0$ then

$$
\int e^{i\lambda \phi(x)} \psi(x) dx \approx \lambda^{-\frac{n}{2}} \sum_{j=0}^{\infty} a_j \lambda^{-j}
$$

where the asymptotics hold in the same sense as in Proposition 3.

Note. The full argument provides a way to compute each coefficient $a_j$ in terms of finitely many derivatives of the functions involved. For instance

$$
a_0 = \psi(x_0)(2\pi)^{\frac{n}{2}} \prod_{j=1}^{n} (-i\mu_j)^{-\frac{1}{2}},
$$
where $\mu_j$ are the eigenvalues of the Hessian of $\phi$. Compare this with the result in 1D.

The idea of the proof is to use the following variant of Morse’s lemma: there is a diffeomorphism from a small neighborhood of $x_0$ to a small neighborhood of the origin under which $\phi$ is transformed into

$$\sum_{j=1}^{k} y_j^2 - \sum_{j=k+1}^{m} y_j^2$$

for some $0 < k \leq n$. Then one emulates a similar argument to the used in 1D. This is easily seen in the case

$$\tilde{\psi}(x) = \Pi_{i=1}^{n} \psi_i(x_i).$$

where $\tilde{\psi}$ is $\psi$ composed with the above diffeomorphism. For the general case, one uses the Taylor expansion for $\tilde{\psi}$ in a similar manner to the proof in 1D.

3. Fourier transforms of measures supported on surfaces.

A fundamental question in harmonic analysis is the decay of the Fourier transform of measure supported on surfaces,

$$\widehat{d\mu}(\xi) = \int_{S} e^{-2\pi i \cdot x} \psi \, d\sigma(x)$$

where we assume the measure is given by $d\mu(x) = \psi \, d\sigma(x)$.

By decay one means having an inequality of type $|\widehat{d\mu}(\xi)| \lesssim |\xi|^{-a}$ for some $a > 0$. If $S$ is simply a hyperplane, say $x_n = 0$ then

$$\widehat{d\mu}(\xi) = \hat{\psi}(\xi_1, ..., \xi_{n-1})$$

and no decay is expected in the $\xi_n$ direction. This is due to the flatness of $S$ with respect to the normal directions to $x_n$.

Now let us consider a different setup. Assume the surface $S$ of dimension $n - 1$ has non-zero Gaussian curvature at each point. By rotating and translating the surface we can write $S$ as unions of pieces as described below. Let $S$ be given by $x_n = \phi(x_1, ..., x_{n-1})$ with $\phi(0) = 0$, $\nabla \psi(0) = 0$, where $\phi$ is smooth. Also we consider the Hessian matrix of $\phi$,

$$[\frac{\partial^2 \phi}{\partial x_i \partial x_j}]_{i,j=1,n}(x_0).$$

Its eigenvalues are the principal curvatures of $S$ and their product is the Gaussian curvature of $S$ at $x_0$ (this also equals the determinant of the Hessian). We have the following result.

**Theorem 1.** Suppose $S$ is a smooth hypersurface in $\mathbb{R}^n$ whose Gaussian curvature is nonzero everywhere and let $d\mu = \psi \, d\sigma$ be as above. Then

$$|\widehat{d\mu}(\xi)| \lesssim |\xi|^{\frac{1-n}{2}}.$$
The proof goes as follows. As explained above, one can reduce the problem to the case when $S$ is given by $x_n = \phi(x_1, ..., x_{n-1})$ with $\phi(0) = 0$, $\nabla \psi(0) = 0$. In this case $d\sigma(x) = \sqrt{1 + |\nabla \phi|^2} dx_1 ... dx_{n-1}$. Also $\psi$ is supported in a small neighborhood of the origin. With $\lambda = |\xi| > 0$ and $\xi = \lambda \eta$ we need to prove

$$\left| \int_{\mathbb{R}^{n-1}} e^{i\lambda \Phi(x, \eta)} \tilde{\psi}(x) dx \right| \lesssim \lambda^{-\frac{n-1}{2}}$$

where $\Phi(x, \eta) = x \cdot \eta = x_1 \eta_1 + ... + x_{n-1} \eta_{n-1} + \phi(x_1, ..., x_{n-1}) \eta_n$. Note that $\nabla_x \Phi = (\eta_1, \eta_2, ..., \eta_{n-1}) + \eta_n \nabla \phi$. We need to deal with three cases $\eta$ close to $(0, ..., 0, 1)$, $\eta$ close to $(0, ..., 0, -1)$ and $\eta$ away from either of these two points.

If $\eta = (0, ..., 0, 1)$, then there is a unique $x \in \mathbb{R}^{n-1}$ such that $\nabla_x \Phi(x) = 0$, precisely $x = 0$. In fact for any $\eta$ near $(0, ..., 0, 1)$, there is a unique $x(\eta)$ (near 0) such that $\nabla \phi(x(\eta), \eta) = 0$. By the implicit function theorem this follows from the fact that the determinant of the Hessian matrix near 0 is not zero, which we know. Thus we can now apply the result from the previous section to obtain the desired decay.

A similar argument is used for $\eta$ close to $(0, ..., 0, -1)$.

Now we consider the third case. Here we have that $\nabla_x \Phi(x, \eta) = (\eta_1, ..., \eta_{n-1}) + \eta_n \nabla \phi(x)$. We also have that $|(|\eta_1, ..., \eta_{n-1}|) \geq c > 0$ and $|\nabla \phi(x)| \lesssim |x|$, hence in a small neighborhood of 0 we have that $|\nabla \Phi| > 0$ and we can obtain the stronger decay.

A classical example of surface with non-vanishing Gaussian curvature is the unit sphere $S^{n-1} \subset \mathbb{R}^n$. As an application of this theory we discuss the spherical maximal function, see Tao’s notes.