Final exam practice

Problem 1. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is differentiable and that \( f'(x) = 1 + 3x + x^4 \), while \( f(0) = 1 \). Find \( f(x) \) and justify that it is the unique solution.

Solution. The function \( F(x) = x + 3\frac{x^2}{2} + \frac{x^5}{5} \) satisfies \( F' = f' \). Since \( \mathbb{R} \) is an interval, it follows that \( f - F = c \), or \( f(x) = F(x) + c \). Let \( x = 0 \), then \( 1 = f(0) = F(0) + c = c \), hence \( f(x) = 1 + x + 3\frac{x^2}{2} + \frac{x^5}{5} \). This is the unique solution since, if \( f' = g' \) and \( f(0) = g(0) \) then \( f = g \).

Problem 2. (T) i) Prove that if \( f : I \to \mathbb{R} \) is differentiable and \( |f'(x)| \leq K, \forall x \in I \) then \( f \) is uniformly continuous on \( I \). Here \( I \) is an interval.

ii) Prove that \( f(x) = \sqrt{x} \) is continuous on \( [0, \infty) \) using the \( \epsilon - \delta \) definition.

iii) Prove that \( f(x) = \sqrt{x} \) is differentiable on \( (0, \infty) \) and compute \( f'(x) \).

iv) Prove that \( f(x) = \sqrt{x} \) is uniformly continuous on \( (1, \infty) \).

v) Prove that \( f(x) = \sqrt{x} \) is uniformly continuous on \( [0, a] \) for any \( a \geq 0 \), but \( f' \) is not bounded on \( (0, a] \).

vi) Prove that \( f(x) = \sqrt{x} \) is uniformly continuous on \( [0, \infty) \).

Solution. i) By Mean Value Theorem it follows that for any \( x, y \in I, x < y \), there is \( c \in (x, y) \) such that \( f(x) - f(y) = f'(c)(x - y) \). Therefore

\[
|f(x) - f(y)| = |f'(c)(x - y)| = |f'(c)||x - y| \leq K|x - y|.
\]

Given \( \epsilon > 0 \), let \( \delta = \frac{\epsilon}{K} \). If \( |x - y| < \delta \), then \( |f(x) - f(y)| \leq K|x - y| < K\delta = K\frac{\epsilon}{K} = \epsilon \). Thus \( f \) is uniformly continuous.

ii) Let \( \epsilon > 0 \). Define \( \delta = \frac{\epsilon^2}{\sqrt{2}} \); the main point is that \( \sqrt{\delta} < \frac{\epsilon}{\sqrt{2}} \). If \( |x - y| \leq \delta \) and \( \max x, y \geq \delta \), then

\[
|\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{\delta}{\sqrt{\delta}} = \sqrt{\delta} < \epsilon
\]

If \( \max x, y \leq \delta \), then \( x, y \leq \delta \) and

\[
|\sqrt{x} - \sqrt{y}| \leq |\sqrt{x}| + |\sqrt{y}| \leq 2\sqrt{\delta} < \epsilon.
\]

iii) From theory, we know that

\[
f'(x) = \frac{1}{2\sqrt{x}}.
\]

In the actual exam you may need to derive this from the formula giving the derivative of the inverse function.

iv) Since \( |f'(x)| = \frac{1}{2\sqrt{x}} \leq \frac{1}{2} \) for \( x \geq 1 \), we can invoke part i) to claim uniformly continuity.

v) \( f \) is continuous on \( [0, a] \) which is a closed bounded interval, thus \( f \) is uniformly continuous on \( [0, a] \). This implies that \( f \) is uniformly continuous on \( (0, a] \).
However \( f' \) is not bounded since \( f'(\frac{1}{n^2}) = \frac{2}{n^2} \). Indeed, if \( f' \) were bounded, then we would have 
\[ |f'(x)| \leq M, \forall x \in (0, a]. \]
But for \( n \) large enough, that is there is \( N, \frac{1}{n^2} \in (0, a], \forall n \geq N \) and \( f'(\frac{1}{n^2}) = \frac{2}{n^2} < M, \forall n \geq N \) is impossible.

vi) We know that \( f \) is uniformly continuous on \([0, 2]\) by part v) and we know that \( f \) is uniformly continuous on \([1, \infty)\) by part iv) and we need to put this together. Let \( \epsilon > 0 \); there is \( \delta_1 > 0 \) such that 
\[ x, y \in [0, 2], |x - y| \leq \delta_1 \Rightarrow |\sqrt{x} - \sqrt{y}| \leq \epsilon; \]
there is \( \delta_2 > 0 \) such that 
\[ x, y \in [1, \infty), |x - y| \leq \delta_2 \Rightarrow |\sqrt{x} - \sqrt{y}| \leq \epsilon; \]
Let \( \delta = \min\{\delta_1, \delta_2, \frac{1}{2}\} \). If \( x, y \geq 0 \) and \( |x - y| < \delta \leq \frac{1}{2} \), then either \( x, y \in [0, 2] \) or \( x, y \in [1, \infty) \). In each case we use the corresponding property above to show that \( |x - y| \leq \delta \Rightarrow |\sqrt{x} - \sqrt{y}| \leq \epsilon. \)

**Problem 3.** (T) i) Write down the definition of a convergent sequence, that is \( \lim_{n \to \infty} a_n = a \).
   (T) ii) Using the definition i) prove that the limit of a sequence is unique, that is if \( \lim_{n \to \infty} a_n = a \) and \( \lim_{n \to \infty} a_n = b \) then \( a = b \).
   iii) Write down the negation of the definition in i), that is of the fact that \( \{a_n\} \) does not converge to \( a \).
   iv) Prove that \( \{a_n\} \) does not converge to \( a \) if and only if there is some \( \epsilon > 0 \) and a subsequence \( \{a_{n_k}\} \) such that \( |a_{n_k} - a| \geq \epsilon \) for all \( k \).
   v) Prove that a bounded sequence does not converge if and only if it has at least two convergent subsequences that converge to different limits.

Solution. i)-iv) Theory and see also exercise 10, section 2.4.
   v) Assume a sequence does not converge. It is bounded, thus it has a subsequence \( a_{n_k} \) that converges to \( a \). Since \( a_n \) does not converge to \( a \), then, by part iv), there is there is some \( \epsilon > 0 \) and a subsequence \( a_{n_j} \) such that \( |a_{n_j} - a| \geq \epsilon \) for all \( j \). Now \( a_{n_j} \) is bounded as well, thus it has a subsequence \( a_{n_l} \) that converges to a limit \( b \). Since \( |a_{n_l} - a| \geq \epsilon, \forall l \), it thus follow that \( a \neq b \).
   Assume that \( a_n \) has two subsequences that converge to different limits. If \( a_n \) were to converge, then any subsequence of it should converge to the same limit, which would contradict the hypothesis.

**Problem 4.** Let \( f : [0, \infty) \to (0, \infty) \) a continuous and differentiable function. Assume \( f'(x) \leq c < 1, \forall x \in [0, \infty) \). Prove that there is a unique \( \tilde{x} \in [0, \infty) \) such that \( f(\tilde{x}) = \tilde{x} \).

Solution. Let \( g(x) = f(x) - x \). We have \( g(0) = f(0) > 0 \). On the other hand \( f(x) - f(0) = f'(c)x \leq cx, \forall x > 0 \). As a consequence \( g(x) = f(x) - x = f(0) + f(x) - x \leq f(0) + (c-1)x \). Since \( c - 1 < 0 \), we can find \( x_0 \) large, say
$x_0 = \frac{f(0)}{1-c}$ such that $g(x_0) \leq f(0) + (c - 1)x_0 \leq 0$. By the intermediate value theorem, there is $y_0 \in [0, x_0]$ such that $g(y_0) = 0$ and this solves $f(y_0) = y_0$.

Assume there are two numbers $x, y$ such that $f(x) = x$ and $f(y) = y$.

Then

$$|x - y| = |f(x) - f(y)| = |f'(z)(x - y)| \leq c|x - y|$$

and, dividing this by $|x - y| \neq 0$, would give $1 \leq c$, contradiction.

**Problem 5.** Let $A, B \subset \mathbb{R}$ be bounded from above. Define the set $A + B = \{a + b : a \in A, b \in B\}$. Prove that $A + B$ is bounded from above and that

$$\sup(A + B) = \sup A + \sup B.$$ 

Note: you are allowed to use all result about sup of a set.

Solution. Since $a \leq \sup A, \forall a \in A$ and $b \leq \sup B, \forall b \in B$, it follows that $a + b \leq \sup A + \sup B, \forall a \in A, b \in B$. This shows that $\sup A + \sup B$ is an upper bound for $A + B$, hence $\sup(A + B) \leq \sup A + \sup B$.

To show the other inequality, we use the following characterization of the sup:

$$\forall \epsilon > 0, \exists a \in A, \sup A - \epsilon < a \leq \sup A.$$ 

Given $\epsilon > 0, \exists a \in A, \sup A - \epsilon < a \leq \sup A$ and $\exists b \in B, \sup B - \epsilon < b \leq \sup B$. From this $\sup A + \sup B - 2\epsilon < a + b \leq \sup(A + B)$. Since $\epsilon$ is arbitrary, this shows that $\sup A + \sup B \leq \sup(A + B)$.

**NOTE:** The use of (T) indicates a theoretical result, thus you should work them based on definitions only or things that you prove. For the problems without (T) you can use all theoretical results from the course.