Final exam practice

Problem 1. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is differentiable and that \( f'(x) = 1 + 3x + x^4 \), while \( f(0) = 1 \). Find \( f(x) \) and justify that it is the unique solution.

Problem 2. (T) i) Prove that if \( f : I \to \mathbb{R} \) is differentiable and \( |f'(x)| \leq K, \forall x \in I \) then \( f \) is uniformly continuous on \( I \). Here \( I \) is an interval.
   ii) Prove that \( f(x) = \sqrt{x} \) is continuous on \([0, \infty)\) using the \( \epsilon - \delta \) definition.
   iii) Prove that \( f(x) = \sqrt{x} \) is differentiable on \((0, \infty)\) and compute \( f'(x) \).
   Is \( f \) differentiable at \( x = 0 \)?
   iv) Prove that \( f(x) = \sqrt{x} \) is uniformly continuous on \((1, \infty)\).
   v) Prove that \( f(x) = \sqrt{x} \) is uniformly continuous on \([0, a] \) for any \( a \geq 0 \), but \( f' \) is not bounded on \((0, a] \).
   vi) Prove that \( f(x) = \sqrt{x} \) is uniformly continuous on \([0, \infty)\).

Problem 3. (T) i) Write down the definition of a convergent sequence, that is \( \lim_{n \to \infty} a_n = a \).
   (T) ii) Using the definition i) prove that the limit of a sequence is unique, that is if \( \lim_{n \to \infty} a_n = a \) and \( \lim_{n \to \infty} a_n = b \) then \( a = b \).
   iii) Write down the negation of the definition in i), that is of the fact that \( \{a_n\} \) does not converge to \( a \).
   iv) Prove that \( \{a_n\} \) does not converge to \( a \) if and only if there is some \( \epsilon > 0 \) and a subsequence \( \{a_{n_k}\} \) such that \( |a_{n_k} - a| \geq \epsilon \) for all \( k \).
   v) Prove that a bounded sequence does not converge if and only if it has at least two convergent subsequences that converge to different limits.

Problem 4. Let \( f : [0, \infty) \to (0, \infty) \) a continuous and differentiable function. Assume \( f'(x) \leq c < 1, \forall x \in [0, \infty) \). Prove that there is a unique \( \hat{x} \in [0, \infty) \) such that \( f(\hat{x}) = \hat{x} \).

Problem 5. Let \( A, B \subset \mathbb{R} \) be bounded from above. Define the set \( A + B = \{a + b : a \in A, b \in B\} \). Prove that \( A + B \) is bounded from above and that
\[
\sup(A + B) = \sup A + \sup B.
\]
Note: you are allowed to use all result about \( \sup \) of a set.

**NOTE:** the use of (T) indicates a theoretical result, thus you should work them based on definitions only or things that you prove. For the problems without (T) you can use all theoretical results from the course.