Practice Midterm 2

**Problem 1.** Let \( a_1 = 1 \) and define recursively
\[
a_{n+1} = \frac{1 + a_n}{2 + a_n}.
\]
Prove that \( \{a_n\} \) is monotone and bounded. Find \( \lim_{n \to \infty} a_n \).

Sol. This is example 2.31 from textbook.

**Problem 2.** Prove that there is a solution of the equation:
\[
x^5 + x^2 = -3, \ x \in \mathbb{R}.
\]

Sol. We consider the function \( f(x) = x^5 + x^2 \) and that it is continuous and that \( f(0) = 0 > -3, f(-2) = -32 + 4 = -28 < -3 \). By the Intermediate Value Theorem, it follows that there is \( x_0 \in (-2,0) \) such that \( f(x_0) = -3 \); that is the solution to our equation.

**Problem 3.** i) Define the continuity of a function \( f : D \to \mathbb{R} \) at \( x_0 \in \mathbb{R} \) both ways: using sequences and using the \( \epsilon - \delta \) language.

ii) Do the same for the uniform continuity.

iii) Assume that \( f : D \to \mathbb{R} \) is Lipschitz, that is there is \( C > 0 \) such that
\[
|f(x) - f(y)| \leq C|x - y|, \ \forall x, y \in D.
\]
Prove that \( f \) is uniformly continuous.

iv) Prove that the function \( f(x) = x^4 \) is uniformly continuous on the interval \([0,2] \) - provide a complete proof, do not use a theoretical result.

v) Prove that the function \( f(x) = x^4 \) is not uniformly continuous on \( \mathbb{R} \).

Sol. i) see textbook.

ii) same

iii) Given any \( \epsilon > 0 \), define \( \delta = \frac{\epsilon}{C} \); if \( C = 0 \), then let \( \delta = 1 \). It then follows that for every \( x, y \) with \( |x - y| \leq \delta \), we have
\[
|f(x) - f(y)| \leq C \cdot \frac{\epsilon}{C} = \epsilon.
\]

iv) We use part iii):
\[
|f(x) - f(y)| = |(x - y)(x^3 + x^2y + xy^2 + y^3)| = |x - y||x^3 + x^2y + xy^2 + y^3|
\]
\[
= |x - y|(x^3 x^2 y + xy^2 + y^3) \leq |x - y|(8 + 8 + 8 + 8) = 32|x - y|.
\]
Since this holds true for all \( x, y \in [0,2] \), it follows that \( f \) is Lipschitz, thus uniformly continuous by part iii).

v) Let \( x_n = n + \frac{1}{n}, y_n = n \). We have that \( \lim_{n \to \infty} (x_n - y_n) = \lim_{n \to \infty} \frac{1}{n} = 0 \), while
\[
x_n^4 - y_n^4 = (n + \frac{1}{n})^4 - n^4 = 4n^3 \frac{1}{n} + 6n^2 \frac{1}{n^2} + 4n \frac{1}{n^3} + \frac{1}{n^4} = 4n^2 + 6 + \frac{4}{n^2} + \frac{1}{n^4}.
\]
Obviously this sequence diverges to \( \infty \), thus \( f \) is not uniformly continuous.