Solution to Problem 3. We follow the steps of mathematical induction. 

\[ P(1) \text{ is true, since the LSH equals the RHS equals } 1. \]

Next we show that \( P(k) \) implies \( P(k + 1) \). \( P(k) \) is given

\[ \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6} \]

As \( n = k + 1 \),

\[ \sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k + 1)^2 \]

\[ = \frac{k(k+1)(2k+1)}{6} + (k + 1)^2 \]

\[ = \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \]

\[ = \frac{(k+1)(k+2)(2k+3)}{6} \]

So the identity also holds for \( n = k + 1 \), hence \( P(k + 1) \) is true.

Hence by Mathematical Induction, we proved the identity for all natural numbers.

Solution to Problem 6. 

6a) was already done in class.

6b. As in the hint, fix \( m \), define \( S(n) \) to be the statement that \( mn \) is a natural number. We prove this by mathematical induction. Since \( m \times 1 = m \) is a natural number, so \( S(1) \) is true. Suppose \( S(k) \) is also true, that’s to say, \( m \times k \) is a natural number. Notice that \( m \times (k + 1) = m \times k + m \) is again a natural number by part (a), since \( m \times k \) and \( m \) are both natural number. So \( S(k + 1) \) is also true. So \( S(n) \) is true for all \( n \), namely \( m \times n \) is a natural number.

Solution to Problem 1.1.7. Consider the set

\[ A := \{n \in \mathbb{N} : \text{either } n = 1 \text{ or } n - 1 \in \mathbb{N} \} \]

(this is the same set as the one provided in the hint).

Notice that \( A \) gathers some of the natural numbers. Formally we have this as

\[ A \subseteq \mathbb{N} \]

If we try to list some of its elements we notice that: \( 1 \in A \), \( 2 \in A \), \( 3 \in A \), ... . So, it seems that \( A \) is quite nice and collects all of the natural numbers (formally, \( A = \mathbb{N} \)). This is just an intuition, not a proof.

Remark 1. Recall that \( \mathbb{N} \) was defined to be the intersection of all inductive subsets of \( \mathbb{R} \). Moreover, \( \mathbb{N} \) is the smallest inductive set in the sense that it’s contained in any other inductive set (equivalently, any other inductive set contains \( \mathbb{N} \)).

So, if we prove that \( A \) is inductive, it must be that \( A \) is a superset of \( \mathbb{N} \), i.e. we get that

\[ A \supseteq \mathbb{N} \]

This, together with the reverse inclusion, allows us to conclude \( A = \mathbb{N} \).

Proof that \( A \) is inductive: We’ve seen that \( 1 \in A \). Next we show that: \( a \in A \Rightarrow a + 1 \in A \). Fix an arbitrary \( a \in A \). Since \( A \subseteq \mathbb{N} \), \( a > 0 \), so we cannot have \( a + 1 = 1 \). Also, from \( a \in A \) and \( A \subseteq \mathbb{N} \), we have \( a \in \mathbb{N} \) and so \( (a + 1) - 1 = a \in \mathbb{N} \). This shows that \( a + 1 \in A \).

Hence, at this point we have \( A = \mathbb{N} \). Now let (as in the hypothesis of the problem) \( n \in \mathbb{N} \), \( n > 1 \). From \( n \in \mathbb{N} \) and \( \mathbb{N} = A \), we have \( n \in A \). Looking at how \( A \) was defined, since \( n \neq 1 \), it must be that \( n - 1 \in \mathbb{N} \).

Solution to Problem 1.1.8. For \( m \in \mathbb{N} \), consider the statement:

\[ S(m) : \forall n \in \mathbb{N}, n > m \text{ we have } n - m \in \mathbb{N} \]
We prove that $S(m)$ is true for any $m \in \mathbb{N}$ by induction.

**The Base Case.** $S(1)$ is the statement that $\forall n \in \mathbb{N}, n > 1$ we have $n - 1 \in \mathbb{N}$. This is precisely the statement of Problem 1.1.7. Hence $S(1)$ is true.

**The Induction Step.** We assume $S(k)$ is true and we want to derive that $S(k+1)$ is true.

Let $n \in \mathbb{N}$, $n > k + 1$. Then definitely $n > k$ (this allows us to apply $S(k)$) and $n - k > 1$. By $S(k)$, we have $n - k \in \mathbb{N}$. Using again the result of Problem 1.1.7 (applied with $n - k$ instead of $n$), we deduce $(n - k) - 1 \in \mathbb{N}$.

But $(n - k) - 1 = n - (k + 1)$. Therefore $n - (k + 1) \in \mathbb{N}$. We just proved that $S(k+1)$ is true.

By the Principle of Mathematical Induction, $S(m)$ is true for any $m \in \mathbb{N}$, i.e.

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N}, n > m \text{ we have } n - m \in \mathbb{N}$$

**Solution to Problem 1.1.9.** We'll show:

(i) $\forall m, n \in \mathbb{Z} : m + n \in \mathbb{Z}$

(ii) $\forall m, n \in \mathbb{Z} : m - n \in \mathbb{Z}$

(iii) $\forall m, n \in \mathbb{Z} : m \cdot n \in \mathbb{Z}$

**Remark 2.** Recall that $\mathbb{Z} = \{−n : n \in \mathbb{N}\} \cup \{0\} \cup \mathbb{N}$ (where by “$−n$” we understand $(−1) \cdot n$ as an operation in $\mathbb{R}$).

A useful property (which is easy to check) is: $k \in \mathbb{Z} \iff −k \in \mathbb{Z}$

**Proof of (i).** Let $m, n \in \mathbb{Z}$. We distinguish 4 cases.

Case 1: both $m, n$ are positive. Then (i) follows from Problem 1.1.6(a).

Case 2: both $m, n$ are negative. Then $n' := −n$ and $m' = −m$ are positive integers, i.e. are natural numbers.

By Case 1, $m' + n' \in \mathbb{Z}$ and by the Remark above, $−(m' + n') \in \mathbb{Z}$. But $−(m' + n') = (−m') + (−n') = m + n$. So $m + n \in \mathbb{Z}$.

Case 3: one of them is zero, say $m = 0$. Then $m + n = n \in \mathbb{Z}$.

Case 4: one of them is positive and the other one is negative, say $m > 0$ and $n < 0$. Then $m \in \mathbb{N}$ and $n' := −n \in \mathbb{N}$.

Notice that $m + n = m − n'$.

If $m > n'$, then by Problem 1.1.8, $m − n' \in \mathbb{N}$. Since $\mathbb{N} \subset \mathbb{Z}$ and $m + n = m − n'$, we get $m + n \in \mathbb{Z}$.

If $m = n'$, then $m + n = 0 \in \mathbb{Z}$.

If $m < n'$, then $m + n = −(n' − m)$. We can apply again Problem 1.1.8 to get $n' − m \in \mathbb{N}$ and by the Remark above $−(n' − m) \in \mathbb{Z}$. Thus $m + n \in \mathbb{Z}$.

**Proof of (ii).** Write $m − n = m + (−n)$. If $m, n \in \mathbb{Z}$, then $m, −n \in \mathbb{Z}$ and by (i), it follows that $m − n \in \mathbb{Z}$.

**Proof of (iii).** Let $m, n \in \mathbb{Z}$. We distinguish 3 cases.

Case 1: $m$ is positive. Then $m \in \mathbb{N}$. We’ll prove that $m \cdot n \in \mathbb{Z}$ by induction on $m$. Fix $n \in \mathbb{Z}$ and consider the statement:

$$S(m) : m \cdot n \in \mathbb{Z}$$

**The base case.** $1 \cdot n = n \in \mathbb{Z}$, so $S(1)$ is true.

**The induction step.** Assume $S(k)$ is true, i.e. $k \cdot n \in \mathbb{Z}$. Then write $(k + 1) \cdot n = k \cdot n + n \in \mathbb{Z}$ (by (i)). This shows $S(k + 1)$ is true.

By the Principle of Mathematical Induction, $S(m)$ is true, for any $m \in \mathbb{N}$.

Case 2: $m = 0$. Then $m \cdot n = 0 \in \mathbb{Z}$.

Case 3: $m < 0$. Then $m' := −m \in \mathbb{N}$ and $m \cdot n = (−m') \cdot n = m' \cdot (−n)$. In this manner, we reduced the problem to Case 1 with $m'$ instead of $m$ and $−n \in \mathbb{Z}$ instead of $n$.