HOMEWORK #3 – SOLUTIONS

Solution to Problem 2.1.6. Since $a_n \to a$, for every $\varepsilon > 0$ there is an index $N \in \mathbb{N}$ starting from which $a_n \in (a - \varepsilon, a + \varepsilon)$ (look at the comment and picture after the “Definition” on p.26). We are free to choose our epsilon to derive some further properties of $\{a_n\}$. Here for example, we can choose $\varepsilon = \frac{a}{2}$. So there is $N \in \mathbb{N}$ s.t. $a_n \in (\frac{a}{2}, \frac{3a}{2})$ for all $n \geq N$. Since $\frac{a}{2} > 0$, $a_n > 0$ for all $n \geq N$.

Solution to Problem 2.1.12. Firstly, we show by induction that the sequence $\{a_n\}$ given by

$$a_{n+1} = \begin{cases} a_n + \frac{1}{n}, & \text{if } a_n^2 \leq 2 \\ a_n - \frac{1}{n}, & \text{if } a_n^2 > 2 \end{cases}, \quad a_1 = 1$$

has positive terms. Before doing that let’s determine some of the first terms in this sequence: $a_2 = a_1 + \frac{1}{1} = 2$ (since $a_1 < \sqrt{2}$), $a_3 = a_2 - \frac{1}{2} = \frac{3}{2}$ (since $a_2 > \sqrt{2}$), $a_4 = a_3 - \frac{1}{3} = \frac{7}{6} < \sqrt{2}$. We see that $\{a_n\}$ takes values both below and above $\sqrt{2}$.

So, for each $n \in \mathbb{N}$ consider the statement:

$$S(n) : a_n > 0$$

The base case is immediately verified for $n = 1$. For the induction step, assume $S(n)$ is true, i.e. $a_n > 0$. In the case $a_n^2 \leq 2$ clearly $a_n + \frac{1}{n} > 0$. In the case $a_n^2 > 2$, suppose that $a_n - \frac{1}{n} \leq 0$, which implies $a_n \leq \frac{1}{n} \leq 1$ and consequently $a_n^2 < 1 < 2$ which contradicts $a_n^2 > 2$; it remains that $a_n - \frac{1}{n} > 0$. In any case, we obtained $a_{n+1} > 0$, i.e. $S(n+1)$ is true. By the Principle of Induction, we deduce that $S(n)$ is true for any $n \in \mathbb{N}$.

Secondly, we prove

$$P(n) : |a_n - \sqrt{2}| < \frac{2}{n} \iff -\frac{2}{n} < a_n - \sqrt{2} < \frac{2}{n}$$

for every $n \in \mathbb{N}$. Again, the base case is easily verified for $n = 1$. It is also easy to check $n = 2$. Next, assume $P(n)$ is true (for some $n \geq 2$). As above we’ll discuss the two cases:

Case 1. $a_n^2 \leq 2$ which is equivalent (as we now know that $a_n > 0$) with $a_n \leq \sqrt{2}$. Then $a_{n+1} = a_n + \frac{1}{n}$, and adding $\frac{1}{n}$ to the inequalities given by $P(n)$, we obtain:

$$-\frac{2}{n} < a_{n+1} - \sqrt{2} < \frac{2}{n}$$

Out of this, we can use the left inequality (the right one is too weak) and since $-\frac{2}{n+1} \leq -\frac{1}{n}$ for any $n \in \mathbb{N}$, we deduce

$$-\frac{2}{n+1} < a_{n+1} - \sqrt{2}$$

To get the right inequality of $P(n+1)$, we use $a_n \leq \sqrt{2}$:

$$a_n + \frac{1}{n} \leq \sqrt{2} + \frac{1}{n} \Rightarrow a_{n+1} - \sqrt{2} \leq \frac{1}{n} < \frac{2}{n+1}$$

(here, we needed $n \geq 2$) Therefore

$$-\frac{2}{n+1} < a_{n+1} - \sqrt{2} < \frac{2}{n+1},$$

i.e. $P(n+1)$ is true.

Case 2. $a_n^2 > 2$ which is equivalent (since $a_n > 0$) with $a_n > \sqrt{2}$. We mirror the arguments from Case 1. Here, $a_{n+1} = a_n - \frac{1}{n}$ so we add $-\frac{1}{n}$ to the inequalities given by $P(n)$:

$$-\frac{3}{n} < (a_n - \frac{1}{n}) - \sqrt{2} < \frac{3}{n} \iff -\frac{3}{n} < a_{n+1} - \sqrt{2} < \frac{1}{n}$$

Out of this, we can use the right inequality (the left one is too weak) and since $\frac{1}{n} < \frac{2}{n+1}$ for any $n \in \mathbb{N}$, we deduce

$$a_{n+1} - \sqrt{2} < \frac{2}{n+1}$$
To get the left inequality of $P(n + 1)$, we use $a_n > \sqrt{2}$:

$$a_n - \frac{1}{n} > \sqrt{2} - \frac{1}{n} \Rightarrow a_{n+1} - \sqrt{2} > -\frac{1}{n} \geq -\frac{2}{n+1}$$

Therefore

$$-\frac{2}{n+1} < a_{n+1} - \sqrt{2} < \frac{2}{n+1},$$

i.e. $P(n + 1)$ is true.

So, in both cases we derived $P(n + 1)$, provided $P(n)$. By the Induction Principle, $P(n)$ is true for any $n \geq 2$, and since it was explicitly checked for $n = 1$, we conclude that $P(n)$ is true for any $n \in \mathbb{N}$.

**Solution to Problem 2.1.14.** Let’s start by computing some terms:

$s_1 = \frac{1}{2}, s_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}, s_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}, \ldots$

We notice a pattern here, namely $s_n = \frac{n}{n+1}$. We try to prove this formally by induction. But first, notice that we have the following recursion formula:

$$s_{n+1} = s_n + \frac{1}{(n+2)(n+1)}$$

So for each $n \in \mathbb{N}$ consider:

$P(n) : s_n = \frac{n}{n+1}$

The base case is immediately checked since we computed the first term. Next, assume $P(n)$ is true. Then, using (1)

$$s_{n+1} = \frac{n}{n+1} + \frac{1}{(n+2)(n+1)} = \frac{n^2 + 2n + 1}{(n+2)(n+1)} = \frac{n + 1}{n + 2}$$

Hence, $P(n + 1)$ is true.

By the Induction Principle $P(n)$ is true for all $n \in \mathbb{N}$, i.e.

$$s_n = \frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}}$$

which goes to $\frac{1}{1+0}$ as $n \to \infty$.

**2nd Method.** The generic term $\frac{1}{(k+1)k}$ can be written as

$$\frac{1}{(k+1)k} = \frac{1}{k} - \frac{1}{k+1}$$

(you can either use partial fractions or write the numerator $1 = (k+1) - k$ and then distribute the denominator) Then

$$s_n = \sum_{k=1}^{n} \frac{1}{(k+1)k} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

(2)

$$= 1 - \frac{1}{n+1}$$

and consequently

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1 - \lim_{n \to \infty} \frac{1}{n+1} = 1$$

**Solution to Problem 2.1.16.** Write

$$\sqrt{n+1} - \sqrt{n} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

Now, notice that

$$0 \leq \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{\sqrt{n} + \sqrt{n}} = \frac{1}{2\sqrt{n}}, \forall n \in \mathbb{N}$$
Since \( \lim_{n \to \infty} \frac{1}{2\sqrt{n}} = 0 \), by the Squeeze Theorem, we get

\[
\lim_{n \to \infty} \left( \sqrt{n+1} - \sqrt{n} \right) = 0
\]

For (b), we have

\[
(\sqrt{n+1} - \sqrt{n})\sqrt{n} = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + 1} = \frac{1}{\sqrt{1 + \frac{1}{n}} + 1}
\]

whose denominator clearly converges to \( \sqrt{1+1} \), hence

\[
\lim_{n \to \infty} (\sqrt{n+1} - \sqrt{n})\sqrt{n} = \frac{1}{2}
\]

**Remark.** We cannot use the Squeeze Theorem as in (a) with the bounds

\[
0 \leq (\sqrt{n+1} - \sqrt{n})\sqrt{n} \leq \left( \frac{1}{2\sqrt{n}} \right) \sqrt{n}
\]

because the last inequality is too weak.

For (c), we use the computation from (b):

\[
(\sqrt{n+1} - \sqrt{n})n = ((\sqrt{n+1} - \sqrt{n})\sqrt{n}) \sqrt{n} = \frac{\sqrt{n}}{\sqrt{1 + \frac{1}{n}} + 1}
\]

which holds for every \( n \in \mathbb{N} \). Since the numerator goes to \(+\infty\) and the denominator converges to 2, we get

\[
\lim_{n \to \infty} (\sqrt{n+1} - \sqrt{n})n = +\infty
\]

**Solution to Problem 2.1.17.** We start by writing the \( \varepsilon - N \) definitions for the two limits:

\[
(*) \quad \lim_{n \to \infty} a_n = +\infty : \forall c > 0, \exists N = N(c) \in \mathbb{N} \text{ s.t. } a_n > c, \forall n \geq N ;
\]

\[
(**) \quad \lim_{n \to \infty} \frac{1}{a_n} = 0 : \forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N} \text{ s.t. } \left| \frac{1}{a_n} - 0 \right| < \varepsilon, \forall n \geq N .
\]

Since \( a_n > 0 \), we have:

\[
(3) \quad \left| \frac{1}{a_n} - 0 \right| < \varepsilon \iff \frac{1}{a_n} < \varepsilon \iff a_n > \frac{1}{\varepsilon}
\]

We were asked to show \((*) \iff (**))\.

\("\Rightarrow\): Assume \((*)\) is true. In order to prove \((**), \) let \( \varepsilon > 0 \). Set \( c := \frac{1}{\varepsilon} \) and apply \((*)\).

So, there is \( N \in \mathbb{N} \text{ s.t. } a_n > c \iff a_n > \frac{1}{\varepsilon} \text{ for all } n \geq N . \) By the equivalence \((3)\), we get \( \left| \frac{1}{a_n} - 0 \right| < \varepsilon \text{ for all } n \in \mathbb{N} . \)

\("\Leftarrow\): Assume \((**)\) is true. In order to prove \((*)\), let \( c > 0 \). Set \( \varepsilon := \frac{1}{c} \) and apply \((**)\).

So, there is \( N \in \mathbb{N} \text{ s.t. } \left| \frac{1}{a_n} - 0 \right| < \varepsilon \iff a_n > \frac{1}{\varepsilon} = c \text{ for all } n \geq N \text{ (here we used again } (3)) . \) Therefore, we have \( a_n > c \text{ for all } n \in \mathbb{N} . \)
Solution to Problem 2.2.1. (a) false: \((-1)^n\) is bounded but is not convergent.
(b) false: \(\left\{\frac{1}{n}\right\}\) is a sequence of positive numbers with limit 0.
(c) false: \(n^2 + 1 > n\) and \(n \to \infty\).
(d) false: by invoking the sequential density of the rationals for \(\sqrt{2}\), we infer that there exists a sequence of rational numbers converging to \(\sqrt{2}\) which is irrational.
An explicit sequence (you’ll see it again later on) is \(\{a_n\}\) with \(a_n = (1 + \frac{1}{n})^n\) and \(\lim_{n \to \infty} a_n = e \in \mathbb{R} \setminus \mathbb{Q}\).
(e) false: the sequence \(\{\frac{1}{n}\}\) is in \((0, 2)\) but its limit is not.

Solution to Problem 2.2.2. Denote \(S := (-\infty, 0] = \{x \in \mathbb{R} : x \leq 0\}\). Let \(\{x_n\}\) be a convergent sequence in \(S\) (i.e. its terms are all in \(S\)). Say \(x = \lim_{n \to \infty} x_n\). We need to show that \(x \in S\).

Since \(x_n \in S\), we have \(x_n \leq 0\) for every \(n \in \mathbb{N}\). Then \(-x_n \geq 0\) for every \(n \in \mathbb{N}\) and \(\lim_{n \to \infty} (-x_n) = -x\). By Lemma 2.21, we get \(-x \geq 0\), equivalently \(x \leq 0\). Hence, \(x \in S\).

Suggestion: Try to go directly through the analogous proof to Lemma 2.21 to solve this, i.e. try a proof by contradiction. I’ve did something similar for Problem 2.1.6.

Solution to Problem 2.2.3. Let \(a \in \mathbb{R}\). Since \(\mathbb{Q}\) is sequentially dense in \(\mathbb{R}\), there exists \(\{a_n\}\) a sequence in \(\mathbb{Q}\) such that \(a_n \to a\), i.e. \(\lim_{n \to \infty} a_n = a\). If we could find irrational terms \(b_n\) such that \(b_n \to 0\), then \(\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n = a\).

So, the problem is reduced to finding a sequence \(\{b_n\}\) of irrational numbers that converges to 0. Take \(b_n = \frac{1}{n + \pi}\). Is easy to check that \(0 < b_n < \frac{1}{n}\) and so \(b_n \to 0\). We know that \(\pi\) is irrational. If we suppose \(\frac{1}{n + \pi}\) is rational, then \(n + \pi\) is rational, and further \(\pi = (n + \pi) - n\) is rational. Therefore, \(\{b_n\}\) is a sequence of irrational numbers, which satisfies our requirements.

Remark. You can imagine other choices for \(\{b_n\}\). For example, \(\{\frac{1}{n}\}\), \(\{\frac{1}{\sqrt{n}}\}\), etc. will work equally well. More sophisticated attempts would be \(\left\{\frac{1}{\sqrt{n}}\right\}\) which almost satisfies this requirement; it merely fails because some of its terms are rationals (when \(n\) is a perfect square, \(1/\sqrt{n}\) is rational). You could try to fix this by choosing \(b_n = \frac{1}{\sqrt{n}\sqrt{n + 1}}\), or by excluding the rational terms and reindex the elements you’re left with.

Solution to Problem 2.2.4. Take the sequence \(\{b_n\}\) from the previous problem, e.g. \(b_n = \frac{1}{n + \pi}\).
Then \(b_n \in \mathbb{R} \setminus \mathbb{Q}\), but \(\lim_{n \to \infty} b_n = 0 \notin \mathbb{R} \setminus \mathbb{Q}\).