HOMEWORK #7 – SOLUTIONS

Solution to Problem 3.3.1. (a) False, consider \( f(x) = 1 \). This function is continuous and its image is \{1\}.
(b) False, consider \( f(x) = 0 \) when \( 0 \leq x < 1 \) and \( f(x) = 1 \) when \( x = 1 \). Its image is \{0\} \( \cup \{1\} \).
(c) False, consider the domain \( D = (1, 2) \cup (3, 4) \) and the function \( f(x) = 1 \) when \( 1 < x < 2 \) and \( f(x) = 2 \) when \( 3 < x < 4 \). This function is continuous and its image is \{1\} \( \cup \{2\} \).
(d) False, consider \( f : [0, 1] \rightarrow \mathbb{R} \) defined by \( f(x) = x(1-x) \). Then, \([f(0), f(1)] = \{0\} \) but we can see that the image of the function is not a single value.

Solution to Problem 3.3.3. Denote by \( f(x) \) the left hand side of the equation. On \((0, \infty)\) this function is continuous (try to justify this for yourself). We can evaluate at \( x = 1 \) and at \( x = 2: f(1) = \frac{1}{\sqrt{2}} - 1 < 0 \) and \( f(2) = \frac{1}{\sqrt{6}} > 0 \). By IVT, there exists \( x_0 \in (1, 2) \) s.t. \( f(x_0) = 0 \).

Solution to Problem 3.3.4. Consider the function \( g(x) = f(x) - x \). Since \( f \) and the identity function are continuous, \( g \) is continuous on \([-1, 1]\). We evaluate: \( g(-1) = f(-1) + 1 \geq -1 + 1 = 0 \) and \( g(1) = f(1) - 1 < 1 - 1 = 0 \). Therefore \( g(-1)g(1) < 0 \), and by IVT there exists \( x_0 \in (-1, 1) \) s.t. \( g(x_0) = 0 \Leftrightarrow f(x_0) = x_0 \). So \( x_0 \) is a fixed point of \( f \).

Solution to Problem 3.3.5. Define a new function \( f(x) := h(x) - g(x) \). Since \( h \) and \( g \) are continuous on \([a, b]\), so \( f \) is continuous on \([a, b]\) as well. Since \( f(a) \leq 0 \) and \( f(b) \geq 0 \), we can apply IVT to conclude that there is \( c \in [a, b] \) such that \( f(c) = 0 \).

Solution to Problem 3.3.6. As in Problem 3.3.4, consider the function \( g(x) = f(x) - x \). Since \( f \) and the identity function are continuous, \( g \) is continuous on \( \mathbb{R} \). Since \( f \) is bounded, there exist \( m, M \in \mathbb{R} \) s.t. \( m \leq f(x) \leq M \) for all \( x \in \mathbb{R} \). Then
\[
g(m-1) = f(m-1) - (m-1) \geq m - m + 1 = 1 > 0 \quad \text{and} \quad g(M+1) = f(M+1) - (M+1) \leq M - M - 1 = -1 < 0 \]
Therefore \( g(m-1)g(M+1) < 0 \). By IVT, there is \( x_0 \in (m-1, M+1) \) s.t. \( g(x_0) = 0 \Leftrightarrow f(x_0) = x_0 \). So \( f \) has at least one fixed point in \( \mathbb{R} \).

Solution to Problem 3.3.10. Suppose \( f \) is not a constant function. Then its image has at least two (distinct) points. Say \( a, b \in f(\mathbb{R}) \) with \( a < b \). By the problem’s hypothesis, \( a, b \) are rational numbers. Since \( \mathbb{R} \setminus \mathbb{Q} \) is dense in \( \mathbb{R} \), there exists \( c \in (a, b) \cap (\mathbb{R} \setminus \mathbb{Q}) \). Since \( f \) is continuous, \( f \) assumes every intermediate value between \( a \) and \( b \) at least once. In particular, there is some \( x_0 \in [0, 1] \) s.t. \( f(x_0) = c \). So \( c \) is a functional value, hence it is rational. Contradiction.

Solution to Problem 3.4.1. (a) Not necessarily true: \( f(x) = x^2 \) is continuous, but not uniformly continuous.
(b) Not necessarily true: \( f(x) = \frac{1}{x} \) is continuous but is not uniformly continuous (it spikes up to infinity at 1). To justify this, take the sequences of general terms: \( u_n := 1 - \frac{1}{n}, v_n = 1 - \frac{2}{n} \). Then check that \( \lim_{n \to \infty} (u_n - v_n) = 0 \) but \( \lim_{n \to \infty} (f(u_n) - f(v_n)) = \infty \).
(c) True: by Theorem 3.17
(d) True: let \( \{u_n\} \) be a convergent sequence to \( u_0 \in D \) and set \( v_n = u_0 \) for all \( n \in \mathbb{N} \). Then \( \lim_{n \to \infty} (u_n - v_n) = 0 \) and this implies that \( \lim_{n \to \infty} (f(u_n) - f(u_0)) = 0 \), so \( \{f(u_n)\} \) converges to \( f(u_0) \).

Solution to Problem 3.4.3. Let \( u_n \) and \( v_n \) be two sequences such that \( \lim_{n \to \infty} [u_n - v_n] = 0 \). Since \( f \) and \( g \) are uniformly continuous, so \( \lim_{n \to \infty} [f(u_n) - f(v_n)] = 0 \) and \( \lim_{n \to \infty} [g(u_n) - g(v_n)] = 0 \). We can add these two limits to obtain
\[
\lim_{n \to \infty} [f(u_n) + g(u_n) - f(v_n) - g(v_n)] = \lim_{n \to \infty} [(f + g)(u_n) - (f + g)(v_n)] = 0
\]
Thus, \( f + g \) is uniformly continuous.
Solution to Problem 3.4.4. Let \( u_n \) and \( v_n \) be two sequences such that \( \lim_{n \to \infty} [u_n - v_n] = 0 \). Then, \( f(u_n) - f(v_n) = (mu_n + b) - (mv_n + b) = m(u_n - v_n) \). Hence,
\[
\lim_{n \to \infty} [f(u_n) - f(v_n)] = \lim_{n \to \infty} [m(u_n - v_n)] = m \left( \lim_{n \to \infty} [u_n - v_n] \right) = 0.
\]

Solution to Problem 3.4.6. Take \( f(x) = x = g(x) \). By Problem 3.4.4, \( f \) is uniformly continuous, but \( h(x) := f(x)f(x) = x^2 \) is not uniformly continuous (see Example 3.15).

Solution to Problem 3.4.7. Let \( M_1, M_2 > 0 \) s.t. \( |f(x)| \leq M_1 \) and \( |g(x)| \leq M_2 \) for all \( x \in D \).

We need to show that \( fg \) is uniformly continuous. Let \( \{u_n\}, \{v_n\} \) be two sequences in \( D \) s.t. \( \lim_{n \to \infty} (u_n - v_n) = 0 \).

Using the hint provided and the triangle inequality, we have:
\[
|f(u_n)g(u_n) - f(v_n)g(v_n)| = |f(u_n) (g(u_n) - g(v_n)) + g(v_n) (f(u_n) - f(v_n))| \\
\leq |f(u_n)| |g(u_n) - g(v_n)| + |g(v_n)| |f(u_n) - f(v_n)| \\
\leq M_1 |g(u_n) - g(v_n)| + M_2 |f(u_n) - f(v_n)|
\]

Now \( f \) and \( g \) are uniformly continuous, so \( |f(u_n) - f(v_n)| \to 0 \) and \( |g(u_n) - g(v_n)| \to 0 \) as \( n \to \infty \). Thus, the right hand side above goes to 0 as \( n \to \infty \). By the Comparison Lemma, we get that
\[
\lim_{n \to \infty} (f(u_n)g(u_n) - f(v_n)g(v_n)) = 0.
\]

It follows that \( f(x)g(x) \) is uniformly continuous on \( D \).

Solution to Problem 3.4.10 We use the \( \varepsilon - \delta \) criterion for \( \varepsilon = 1 \). So there exists a \( \delta > 0 \) s.t. if \( u, v \in (a, b) \) with \( |u - v| < \delta \), then \( |f(u) - f(v)| < 1 \). So, for any \( u_0 \in (a, b) \) and any interval of length less than \( \delta \) of the form \((\alpha, u_0]\), we have that \( |u - u_0| < \delta \) and therefore:
\[
|f(u) - f(u_0)| < 1 \Rightarrow 1 > |f(u) - f(u_0)| \geq |f(u) - f(u_0)| \Rightarrow |f(u)| < 1 + |f(u_0)|
\]

The idea is to chop up the interval \((a, b)\) in smaller intervals of size \( < \delta/2 \). Now let \( n \in \mathbb{N} \) be s.t. \( n > \frac{b-a}{\delta/2} \) and consider the intermediate points: \( a_0 := a, a_1 := a + \frac{b-a}{n}, a_2 := a + 2 \frac{b-a}{n}, \ldots, a_n := a + n \frac{b-a}{n} = b \). Notice that we can cover the interval \((a, b)\) by taking the unions of all intervals \((a_{k-1}, a_{k+1})\), \( 1 \leq k \leq n - 1 \). Also \( a_k \in (a_{k-1}, a_{k+1}) \) and for any \( u \in (a_{k-1}, a_{k+1}) \) we have \( |u - a_k| \leq \frac{\delta}{2} < \delta \). Therefore for any \( u \in (a, b) \), there exists \( k \in \{1, 2, \ldots, n-1\} \) s.t. \( u \) belongs to \((a_{k-1}, a_{k+1})\) and thus \( |f(u)| < 1 + |f(a_k)| \). Take \( M := \max\{|1 + |f(a_1)|, 1 + |f(a_2)|, \ldots, 1 + |f(a_{n-1})|\} \).

So, we get that for all \( u \in (a, b) \), \( |f(u)| \leq M \). This shows that \( f \) is bounded.

Solution to Problem 3.4.11 We’ll check the definition of uniform continuity. Let \( \{u_n\}, \{v_n\} \) be two sequences in \((a, b)\) s.t. \( \lim_{n \to \infty} (u_n - v_n) = 0 \). By the Lipschitz continuity property we have
\[
|f(u_n) - f(v_n)| \leq C |u_n - v_n|, \quad \forall n \in \mathbb{N}
\]

Using the Comparison Lemma, we deduce that \( \lim_{n \to \infty} (f(u_n) - f(v_n)) = 0 \).

It follows that \( f \) is uniformly continuous.

Solution to Problem 3.5.2. First, we will show the inequality. We will split this into two cases in order to eliminate the absolute values on both sides of the inequality.

- First case, \( x > x_0 \): Then, the inequality becomes
\[
\sqrt{x} - \sqrt{x_0} \leq \frac{x - x_0}{\sqrt{x_0}} \iff \sqrt{x x_0} \leq x \iff \sqrt{x} \leq \sqrt{x_0}
\]
which is true by the assumption.

- Second case, \( x \leq x_0 \): Then, the inequality becomes
\[
\sqrt{x_0} - \sqrt{x} \leq \frac{x_0 - x}{\sqrt{x_0}} \iff x \leq \sqrt{x x_0} \iff \sqrt{x} \leq \sqrt{x_0}
\]
which is also true by the assumption.
Now we prove the continuity. Let $\varepsilon > 0$ be arbitrary. By choosing $\delta = \varepsilon \sqrt{x_0}$, we have that

$$|x - x_0| < \varepsilon \sqrt{x_0} \quad \text{for } x_0 = 4 \text{ or } x_0 = 100$$

Therefore, from the inequality above,

$$|\sqrt{x} - \sqrt{x_0}| \leq |x - x_0|/\sqrt{x_0} < \frac{\varepsilon \sqrt{x_0}}{\sqrt{x_0}} = \varepsilon \quad \text{for } x_0 = 4 \text{ or } x_0 = 100.$$

We conclude that $f(x) = \sqrt{x}$ is continuous at $x_0 = 4$ and $x_0 = 100$.

**Solution to Problem 3.5.3.** Let $\varepsilon > 0$ be arbitrary. Assume that $|x - x_0| < \delta$ where $\delta$ will be defined later so that $|x^3 - x_0^3| < \varepsilon$. Note that

$$|x^3 - x_0^3| = |x - x_0||x^2 + xx_0 + x_0^2|$$

The common trick here is to first assume that $\delta \leq 1$, which implies $|x - x_0| < 1$. By the triangle inequality,

$$|x| - |x_0| \leq |x - x_0| < 1 \implies |x| < |x_0| + 1$$

Thus, we can bound the last term in (1) above by a function of $x_0$

$$|x^2 + xx_0 + x_0^2| \leq |x|^2 + |x||x_0| + |x_0|^2 < (|x_0| + 1)^2 + (|x_0| + 1)|x_0| + |x_0|^2 = 3|x_0|^2 + 3|x_0| + 1$$

Consequently, if we choose $\delta$ such that $\delta \leq \frac{\varepsilon}{3|x_0|^2 + 3|x_0| + 1}$, then from (1) we have

$$|x^3 - x_0^3| < \frac{\varepsilon}{3|x_0|^2 + 3|x_0| + 1} \cdot (3|x_0|^2 + 3|x_0| + 1) = \varepsilon.$$

and we conclude that $f(x) = x^3$ is continuous. The proof above requires that $\delta \leq 1$ and $\delta \leq \frac{\varepsilon}{3|x_0|^2 + 3|x_0| + 1}$, so a sensible choice of $\delta$ here is $\delta = \min \left\{ 1, \frac{\varepsilon}{3|x_0|^2 + 3|x_0| + 1} \right\}$.

**Solution to Problem 3.5.4.** Start by plotting the graph of $f$. Notice that $f(3/4) = 7/4 < 2$. Choose an $\varepsilon > 0$ s.t. $f(\frac{3}{4}) + \varepsilon < 2$. For example, take $\varepsilon = \frac{2 - \frac{3}{4}}{2} = \frac{1}{8}$. In this way $(f(\frac{3}{4}) - \varepsilon, f(\frac{3}{4}) + \varepsilon) \subset (-\infty, 2)$. No matter how small we choose $\delta > 0$, $\delta + \frac{3}{4} \in (\frac{3}{4} - \frac{3}{4}, \frac{3}{4} + \delta)$ and $f(\frac{3}{4} + \frac{3}{4}) = 2 \notin (f(\frac{3}{4}) - \varepsilon, f(\frac{3}{4}) + \varepsilon)$. So by the $\varepsilon - \delta$ criterion, $f$ is not continuous at $x_0 = \frac{3}{4}$.

**Solution to Problem 3.5.5.** Let $x_0$ in $\mathbb{R}$ and $\varepsilon > 0$. We need to show:

$$\exists \delta > 0 : |x - x_0| < \delta \implies |h(x) - h(x_0)| < \varepsilon$$

So we need to produce a $\delta$ (will depend on $\varepsilon$, but it may also depend on $x_0$; if you give one which is independent of $x_0$ you prove a stronger statement, namely that $h$ is uniformly continuous) such that if $x \in (x_0 - \delta, x_0 + \delta)$, then $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

1st Method: With a $\delta > 0$ to be chosen later, we have:

$$|h(x) - h(x_0)| = \left| \frac{1}{1 + x^2} - \frac{1}{1 + x_0^2} \right| = \left| \frac{x^2 - x_0^2}{(1 + x^2)(1 + x_0^2)} \right| = \left| \frac{x^2 - x_0^2}{(1 + x^2)(1 + x_0^2)} \right| = |x - x_0| \cdot \frac{|x + x_0|}{(1 + x^2)(1 + x_0^2)}$$

Since $1 + x^2 \geq 1, 1 + x_0^2 \geq 1$ and $|x + x_0| \leq |x| + |x_0|$, we can write down

$$|h(x) - h(x_0)| \leq |x - x_0| \cdot \frac{|x| + |x_0|}{1} \leq |x - x_0| \cdot (|x - x_0| + |x_0|)$$

(we used $|x| = |(x - x_0) + x_0| \leq |x - x_0| + |x_0|$. Therefore

$$|h(x) - h(x_0)| < \delta(\delta + x_0)$$

We can now choose $\delta$ so that $\delta(\delta + x_0) < \varepsilon$. For example $\delta := \min\{1, \frac{\varepsilon}{2(1 + |x_0|)}\}$ (is $> 0$) satisfies

$$\delta(\delta + |x_0|) \leq \frac{\varepsilon}{2(1 + |x_0|)} \cdot (1 + |x_0|) = \frac{\varepsilon}{2} < \varepsilon$$

Thus if $x$ is s.t. $|x - x_0| < \delta$ then $|h(x) - h(x_0)| < \delta(\delta + x_0) < \varepsilon$. 


2nd Method: As above we have 

\[ |h(x) - h(x_0)| = |x - x_0| \cdot \frac{|x + x_0|}{(1 + x^2)(1 + x_0^2)} \]

The above fraction can be estimated as follows:

\[ \frac{|x + x_0|}{(1 + x^2)(1 + x_0^2)} \leq \frac{|x| + |x_0|}{(1 + x^2)(1 + x_0^2)} = \frac{|x|}{(1 + x^2)(1 + x_0^2)} + \frac{|x_0|}{(1 + x^2)(1 + x_0^2)} \leq \frac{|x|}{(1 + x^2) \cdot 1} + \frac{|x_0|}{1 \cdot (1 + x_0^2)} \]

Here we notice that for every \( x \in \mathbb{R} \), we have \( 2|x| \leq x^2 + 1 \iff \frac{|x|}{1 + x^2} \leq \frac{1}{2} \). Therefore

\[ \frac{|x + x_0|}{(1 + x^2)(1 + x_0^2)} \leq \frac{1}{2} + \frac{1}{2} = 1 \]

and consequently

\[ |h(x) - h(x_0)| \leq |x - x_0| \]

Here we notice that we can choose \( \delta = \varepsilon \) (independent of \( x_0 \)) s.t. if \( |x - x_0| < \delta \) then \( |h(x) - h(x_0)| < \delta = \varepsilon \).

3rd Method: We write \( h = f \circ p \), where \( f : (0, \infty) \to (0, \infty) \), \( f(y) = \frac{1}{y} \) and \( p : \mathbb{R} \to (0, \infty) \), \( p(x) = x^2 + 1 \). Since \( p(x) \) is a polynomial, it is continuous (see Corollary in the textbook). If we show that \( f \) is continuous on \((0, \infty)\) then it follows that \( h \) is continuous (being a composition of two continuous functions).

So let \( x_0 > 0 \) and \( \varepsilon > 0 \). We have:

\[ |f(x) - f(x_0)| = \left| \frac{1}{x} - \frac{1}{x_0} \right| = \frac{|x - x_0|}{|x_0x|} = \frac{|x - x_0|}{|x_0|} \]

If we’ll choose \( \delta \leq \frac{x_0}{2} \), then if \( |x - x_0| < \delta \), we get \( x \in (x_0 - \delta, x_0 + \delta) \subset (\frac{x_0}{2}, \frac{3x_0}{2}) \) and then

\[ \frac{1}{|x||x_0|} = \frac{1}{x_0} < \frac{1}{\frac{x_0}{2}} = \frac{2}{x_0} \]

So, choosing \( \delta \leq \varepsilon \frac{x_0^2}{4} \), for \( |x - x_0| < \delta \) we can estimate

\[ |f(x) - f(x_0)| = |x - x_0| \cdot \frac{1}{|x_0|} < \delta \cdot \frac{2}{x_0} \leq \varepsilon \cdot 1 \]

We have two conditions on \( \delta \) above. Choosing \( \delta := \min\{\frac{x_0}{2}, \varepsilon \frac{x_0^2}{4}\} \) allows us to make use of all the inequalities above. Thus, if \( |x - x_0| < \delta \) it must be that \( |f(x) - f(x_0)| < \varepsilon \). It follows that \( f \) is continuous.

Solution to Problem 3.5.6. Let \( \varepsilon > 0 \). Take \( \delta := \frac{\varepsilon}{C} > 0 \). Then if \( u, v \in D \) are such that \( |u - v| < \delta \), then

\[ |f(u) - f(v)| \leq C|u - v| < C\delta = C\frac{\varepsilon}{C} = \varepsilon \cdot 1 \]

So \( f \) verifies the \( \varepsilon - \delta \) criterion on \( D \).

Solution to Problem 3.5.7.(a) Let \( x_0 \in [0, 1] \). Let \( \varepsilon > 0 \). We have

\[ |\sqrt{x} - \sqrt{x_0}| = \frac{|(\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})|}{|\sqrt{x} + \sqrt{x_0}|} = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} \leq \frac{1}{\sqrt{x}} \cdot |x - x_0| \]

If \( x_0 > 0 \), we can take \( \delta = \min\{\frac{x_0}{2}, \sqrt{x_0} \cdot \varepsilon\} \) (which is \( > 0 \)) and if \( x \in (x_0 - \delta, x_0 + \delta) \), then \( x \geq \frac{x_0}{2} \) we deduce

\[ |f(x) - f(x_0)| = |\sqrt{x} - \sqrt{x_0}| < \frac{1}{\sqrt{x_0}} \cdot \delta \leq \frac{1}{\frac{x_0}{2}} \cdot \sqrt{x_0} \cdot \varepsilon = \varepsilon \]

If \( x_0 = 0 \), then we can take \( \delta = \varepsilon^2 \) to obtain that \( x \in (-\delta, \delta) \cap [0, 1] \) implies

\[ |f(x) - f(0)| = \sqrt{x} < \sqrt{\delta} = \varepsilon \]

Therefore for every \( \varepsilon > 0 \), there exists some \( \delta > 0 \) so that if \( x \in (x_0 - \delta, x_0 + \delta) \cap [0, 1] \) then \( |f(x) - f(x_0)| < \varepsilon \). Hence, \( f \) is continuous at \( x_0 \). Since \( x_0 \) was arbitrarily taken in \([0, 1]\), we have that \( f \) is continuous on \([0, 1]\).

(b) \( f \) is continuous on the bounded and closed interval \([0, 1]\), so by Theorem 3.17, it is uniformly continuous.

(c) Suppose that \( f \) is a Lipschitz function (\( \Leftrightarrow \) Lipschitz continuous). So there exists a constant \( C > 0 \) s.t.

\[ |f(u) - f(v)| \leq C|u - v| , \quad \forall u, v \in [0, 1] \]
So for all \( u, v \in [0, 1], u \neq v \) we have
\[
\left| \frac{f(u) - f(v)}{u - v} \right| \leq C
\]
But
\[
\frac{f(u) - f(v)}{u - v} = \frac{\sqrt{u} - \sqrt{v}}{(\sqrt{u} - \sqrt{v})(\sqrt{u} + \sqrt{v})} = \frac{1}{\sqrt{u} + \sqrt{v}}
\]
Take \( v = 0 \) and \( u_n = \frac{1}{n^2} \). Clearly \( v, u_n \in [0, 1] \) for all \( n \in \mathbb{N} \) and so
\[
C \geq \left| \frac{f(u_n) - f(v)}{u_n - v} \right| = \frac{1}{n} = n
\]
Hence, for all \( n \in \mathbb{N} \) we have \( n \leq C \). This contradicts the fact that \( \mathbb{N} \) is unbounded. Therefore, \( f \) is not a Lipschitz function.