Midterm 1

Problem 1. (14 points) Let $f : [0, 1] \rightarrow \mathbb{R}$ be bounded.
   i) (2 points) Write down the definition of a partition $P$ of $[a, b]$.
   ii) (2 points) Write down the definition of $L(f, P), U(f, P)$.
   iii) (2 points) Write down the definition of $\int_a^b f, \int_a^b f$.
   iv) (4 points) Prove that $\int_a^b f \leq \int_a^b f$.
   v) (2 points) Write down the definition of integrability of $f$.
   vi) (2 points) Write down the alternative characterization of integrability using an Archimedean sequence of partitions.

Solution. All the definitions are provided in textbook. Part iv) is Lemma 6.4, see the proof there.
Problem 2. (12 points) Let \( f : [0, 1] \to \mathbb{R} \) defined by \( f(x) = x \). Prove that \( f \) is integrable and compute \( \int_0^1 x \, dx \) by using only the definition of integrability (as stated in parts v) or vi) in Problem 1).

Hint: use regular partitions.

Solution Let \( P_n = \{ \frac{i}{n}; 0 \leq i \leq n \} \) be the regular partition of \([0, 1]\). On each partition interval \([\frac{i-1}{n}, \frac{i}{n}]\) we have \( m_i = \frac{i-1}{n} \), \( M_i = \frac{i}{n} \) and \( \frac{i}{n} - \frac{i-1}{n} = \frac{1}{n} \). Based on these we compute:

\[
L(f, P_n) = \sum_{i=1}^{n} \frac{i-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} (i-1) = \frac{1}{n^2} \sum_{i=0}^{n-1} i = \frac{1}{n^2} \frac{(n-1)n}{2} = \frac{n-1}{2n}.
\]

\[
U(f, P_n) = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{n+1}{2n}.
\]

From this it follows that

\[
\lim_{n \to \infty} U(f, P_n) - L(f, P_n) = \lim_{n \to \infty} \frac{1}{n} = 0
\]

therefore \( f \) is integrable. Moreover

\[
\int_0^1 f = \lim_{n \to \infty} U(f, P_n) = \lim_{n \to \infty} \frac{n+1}{2n} = \frac{1}{2}.
\]
Problem 3. (12 points) Consider the function $f : [0, 1] \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 2x, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Is $f$ integrable? Prove your claim.

Solution Let $P = \{x_0, x_1, ..., x_n\}$ be any partition. On each partition interval $[x_{i-1}, x_i]$ we have $m_i = x_{i-1}$, $M_i = 2x_i$ (we’ll prove this at the end). Therefore

$$L(f, P) = L(x, P), \quad U(f, P) = U(2x, P).$$

As a consequence we obtain

$$\int_0^1 f = \sup_P \{ L(f, P), P \text{- partition} \} = \sup_P \{ L(x, P), P \text{- partition} \} = \int_0^1 x = \frac{1}{2},$$

$$\int_0^1 f = \inf_P \{ U(f, P), P \text{- partition} \} = \inf_P \{ U(2x, P), P \text{- partition} \} = \int_0^1 2x = 1$$

where we have used the result from Problem 2. Since the lower and upper integral do not equal, $f$ is not integrable.

Now let’s prove that $M_i = 2x_i$. From the definition of $f$ we have $f(x) \leq 2x$, \forall x \in [x_{i-1}, x_i]$. Thus $2x_i$ is an upper bound for the set $\{ f(x) | x \in [x_{i-1}, x_i] \}$.

If $x_i \in \mathbb{R} \setminus \mathbb{Q}$ then $f(x_i) = 2x_i$ and $M_i = 2x_i$.

If $x_i \in \mathbb{Q}$, then for any $\epsilon > 0$, there is $r \in \mathbb{R} \setminus \mathbb{Q}$ and $r \in [x_i - \epsilon, x_i]$ thus $2r - 2\epsilon \geq f(r) = 2r \geq 2x_i$. This shows that $M_i = 2x_i = \sup \{ f(x) | x \in [x_{i-1}, x_i] \}$. 
Problem 4. (12 points) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function with the property that $\int_a^b f = 0$ for all $0 \leq a < b \leq 1$. Prove that $f(x) = 0, \forall x \in [0, 1]$.

Solution We argue by contradiction. Assume that there is $x_0 \in [0, 1]$ such that $f(x_0) \neq 0$. Without restricting generality of the argument assume $f(x_0) > 0$. Using the continuity of $f$, we set $\epsilon = \frac{f(x_0)}{2}$ and obtain that there is $\delta > 0$ such that $|f(x) - f(x_0)| \leq \frac{f(x_0)}{2}$ for all $x \in [x_0 - \delta, x_0 + \delta]$. Therefore $f(x) \geq \frac{f(x_0)}{2}$ for all $x \in [x_0 - \delta, x_0 + \delta]$. From this we conclude with

$$\int_{x_0 - \delta}^{x_0 + \delta} f(x) dx \geq \int_{x_0 - \delta}^{x_0 + \delta} f(x_0) dx = 2\delta f(x_0) > 0$$

contradiction!

Note: in this argument we are not very careful about whether $[x_0 - \delta, x_0 + \delta] \subset [0, 1]$. One can slightly modify the argument to take care of this: if $\delta < \frac{1}{2}$, then $[x_0 - \delta, x_0] \subset [0, 1]$ or $[x_0, x_0 + \delta] \subset [0, 1]$ and then the arguments works the same.