HOMEWORK #6 – SOLUTIONS

Solution to Problem 8.5.2
A quick calculation shows that \((-1)^k = 1\) for non-negative values of \(k\).

Then \((1 + x)^{-1} = \sum_{k=0}^{\infty} \left(-\frac{1}{k}\right) x^k = \sum_{k=0}^{\infty} (-x)^k = (by \text{ the geometric series formula}) \frac{1}{1+x}\)

Solution to Problem 8.5.4. By the extreme value theorem, there is \(x_m, x_M \in [a,b]\) such that \(g(x_m) \leq f(x) \leq g(x_M), \forall x \in [a,b]\). Since \(h \geq 0\) we obtain \(g(x_m)h(x) \leq f(x)h(x) \leq g(x_M)h(x), \forall x \in [a,b]\). From this we obtain

\[
g(x_m) \int_a^b h(x)dx = \int_a^b g(x_m)h(x)dx \leq \int_a^b g(x)h(x)dx \leq \int_a^b g(x_M)h(x)dx = g(x_M) \int_a^b h(x)dx.
\]

Assume \(\int_a^b h(x) \neq 0\), in which case \(\int_a^b h(x) > 0\), since \(h \geq 0\). From the above, we obtain

\[
g(x_m) \leq \frac{1}{\int_a^b h(x)dx} \int_a^b g(x)h(x)dx \leq g(x_M).
\]

Using the intermediate value theorem for \(g\), it follows that there is \(y \in [a,b]\) such that \(g(c) = \frac{1}{\int_a^b h(x)dx} \int_a^b g(x)h(x)dx\) and the conclusion follows.

It is possible that \(\int_a^b h(x) = 0\); since \(h \geq 0\) and it is continuous, we know from Problem 6 in Section 6.4, that in this case \(h(x) = 0, \forall x\). In this case the conclusion of the problem holds with any \(c\) since both sides are 0.

Solution to Problem 8.5.5 The Cauchy reminder theorem states that

\[
f(x) - p_n(x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt.
\]

If \(x > x_0\), then let \(g(t) = f^{(n+1)}(t)\) and \(h(t) = (x-t)^n\). Applying the result from the Problem 4, we obtain that there is \(c \in [x_0, x]\) such that

\[
\frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(c) (x-t)^n dt
\]

\[
= \frac{1}{n!} \int_{x_0}^{x} (x-t)^n \frac{(x-t)^{n+1}}{n+1} dt
\]

\[
= \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(c) \frac{(x-x_0)^{n+1}}{n+1} dt
\]

\[
= \frac{1}{(n+1)!} f^{(n+1)}(c)(x-x_0)^{n+1}.
\]

and this is the result claimed by the Lagrange reminder theorem.

A similar proof can be made if \(x < x_0\).

Solution to Problem 8.5.8
Fix an \(x\) with \(|x| > 1\), and writing \(c_k = \left(\binom{n}{k}\right) x^k\), the Binomial Expansion is \(\sum_{k=0}^{\infty} c_k\).

In the proof of Lemma 8.21, it is calculated that \(\lim_{n \to \infty} \left|\frac{c_k}{c_k}\right| = |x|\), which is greater than 1.

Therefore we can apply Lemma 8.20 to conclude that \(c_k\) is unbounded (ie get arbitrarily large in absolute value).

In particular, \(c_k\) does not go to 0 as \(n \to \infty\). But the terms of any convergent series must go to zero (see Proposition 9.5 for a quick proof of this). Therefore \(\sum_{k=0}^{\infty} c_k\) does not converge.

Solution to Problem 8.6.4.
From the Lagrange reminder theorem we obtain that, for some \(c\) between 0 and \(x\),

\[
g(x) - p_n(x) = \frac{g^{n+1}(c)}{(n+1)!} x^{n+1}.
\]
Using that $g^{n+1}$ is bounded, it follows that
\[ |g(x) - p_n(x)| \leq M|x|^{n+1}. \]

Next,
\[ |p_n(x)| \leq |p_n(x) - g(x)| + |g(x)| \leq M|x|^{n+1} + c_{n+1}|x|^{n+1} = K|x|^{n+1}, \]
for $|x| \leq \delta_n$. With $p_n(x) = a_0 + a_1 x + \ldots + a_n x^n$, this becomes
\[ |a_0 + a_1 x + \ldots + a_n x^n| \leq M|x|^{n+1}. \]

Letting $x = 0$ gives $a_0 = 0$. Then we divide by $x$,
\[ |a_1 + a_2 x + \ldots + a_n x^{n-1}| \leq M|x|^n. \]

for all $|x| \leq \delta_n$ with $x \neq 0$ (since we divided by $x$). Passing to the limit with $x \to 0$, gives $a_1 = 0$; we continue this to obtain $a_0 = a_1 = \ldots = a_n = 0$. Since $a_k = \frac{g^{(k)}(0)}{k!}$, it follows that $g^{(k)}(0) = 0$ for $k = 0, 1, \ldots, n$. But this is true for all $n$. 