Solution to Problem 1.1.2(a) False. The set \((0, 1)\) is bounded above but does not have a largest number.

(b) True. Since 0 is a lower bound of \(S\) and \(\inf S\) is the greatest lower bound, so \(0 \leq \inf S\).

(e) True. \(\sup S\) is an upper bound of \(S\), and so of \(B\). Since \(\sup B\) is the least upper bound of \(B\), so \(\sup B \leq \sup S\).

Solution to Problem 3. We follow the steps of mathematical induction.

\(P(1)\) is true, since the LSH equals the RHS equals 1.

Next we show that \(P(k)\) implies \(P(k + 1)\). \(P(k)\) gives

\[
\sum_{j=1}^{k} j^2 = \frac{k(k + 1)(2k + 1)}{6}
\]

As \(n = k + 1\),

\[
\sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k + 1)^2
= \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2
= \frac{(k + 1)(2k^2 + k + 6k + 6)}{6}
= \frac{(k + 1)(k + 2)(2k + 3)}{6}
\]

So the identity also holds for \(n = k + 1\), hence \(P(k + 1)\) is true.

Hence by Mathematical Induction, we proved the identity for all natural numbers.

Solution to Problem 4. Notice that \(\sum_{j=1}^{n} j(j + 1) = \sum_{j=1}^{n} (j^2 + j) = \sum_{j=1}^{n} j^2 + \sum_{j=1}^{n} j\). We can use the result from Problem 3 for the first term and Example 1.1 for the second term.

Solution to Problem 6. 6a) was already done in class.

6b. As in the hint, fix \(m\), define \(S(n)\) to be the statement that \(mn\) is a natural number. We prove this by mathematical induction. Since \(m \times 1 = m\) is a natural number, so \(S(1)\) is true. Suppose \(S(k)\) is also true, that’s to say, \(m \times k\) is a natural number. Notice that \(m \times (k + 1) = m \times k + m\) is again a natural number by part (a), since \(m \times k\) and \(m\) are both natural number. So \(S(k + 1)\) is also true. So \(S(n)\) is true for all \(n\), namely \(m \times n\) is a natural number.

Solution to Problem 1.1.7. Consider the set

\[
A := \{n \in \mathbb{N} : \text{either } n = 1 \text{ or } n - 1 \in \mathbb{N}\}
\]

(this is the same set as the one provided in the hint).

Notice that \(A\) gathers some of the natural numbers. Formally we have this as

\[
A \subseteq \mathbb{N}
\]

If we try to list some of its elements we notice that: \(1 \in A, 2 \in A, 3 \in A, \ldots\). So, it seems that \(A\) is quite nice and collects all of the natural numbers (formally, \(A = \mathbb{N}\)). This is just an intuition, not a proof.

Remark 1. Recall that \(\mathbb{N}\) was defined to be the intersection of all inductive subsets of \(\mathbb{R}\). Moreover, \(\mathbb{N}\) is the smallest inductive set in the sense that it’s contained in any other inductive set (equivalently, any other inductive set contains \(\mathbb{N}\)).

So, if we prove that \(A\) is inductive, it must be that \(A\) is a superset of \(\mathbb{N}\), i.e. we get that

\[
A \supseteq \mathbb{N}
\]

This, together with the reverse inclusion, allows us to conclude \(A = \mathbb{N}\).
Proof that $A$ is inductive: We’ve seen that $1 \in A$. Next we show that: $a \in A \Rightarrow a + 1 \in A$. Fix an arbitrary $a \in A$. Since $A \subseteq \mathbb{N}$, $a > 0$, so we cannot have $a + 1 = 1$. Also, from $a \in A$ and $A \subseteq \mathbb{N}$, we have $a \in \mathbb{N}$ and so $(a + 1) - 1 = a \in \mathbb{N}$. This shows that $a + 1 \in A$.

Hence, at this point we have $A = \mathbb{N}$. Now let (as in the hypothesis of the problem) $n \in \mathbb{N}$, $n > 1$. From $n \in \mathbb{N}$ and $\mathbb{N} = A$, we have $n \in A$. Looking at how $A$ was defined, since $n \neq 1$, it must be that $n - 1 \in \mathbb{N}$.

Solution to Problem 1.1.8. For $m \in \mathbb{N}$, consider the statement:

$$S(m) : \forall n \in \mathbb{N}, n > m \text{ we have } n - m \in \mathbb{N}$$

We prove that $S(m)$ is true for any $m \in \mathbb{N}$ by induction.

The Base Case. $S(1)$ is the statement that $\forall n \in \mathbb{N}, n > 1$ we have $n - 1 \in \mathbb{N}$. This is precisely the statement of Problem 1.1.7. Hence $S(1)$ is true.

The Induction Step. We assume $S(k)$ is true and we want to derive that $S(k + 1)$ is true.

Let $n \in \mathbb{N}$, $n > k + 1$. Then definitely $n > k$ (this allows us to apply $S(k)$) and $n - k > 1$. By $S(k)$, we have $n - k \in \mathbb{N}$. Using again the result of Problem 1.1.7 (applied with $n - k$ instead of $n$), we deduce $(n - k) - 1 \in \mathbb{N}$.

But $(n - k) - 1 = n - (k + 1)$. Therefore $n - (k + 1) \in \mathbb{N}$. We just proved that $S(k + 1)$ is true.

By the Principle of Mathematical Induction, $S(m)$ is true for any $m \in \mathbb{N}$, i.e.

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N}, n > m \text{ we have } n - m \in \mathbb{N}$$

Solution to Problem 1.1.9. We’ll show:

(i) $\forall m, n \in \mathbb{Z} : m + n \in \mathbb{Z}$
(ii) $\forall m, n \in \mathbb{Z} : m - n \in \mathbb{Z}$
(iii) $\forall m, n \in \mathbb{Z} : m \cdot n \in \mathbb{Z}$

Remark 2. Recall that $\mathbb{Z} = \{-n : n \in \mathbb{N}\} \cup \{0\} \cup \mathbb{N}$ (where by “$-n$” we understand $-1 \cdot n$ as an operation in $\mathbb{R}$). A useful property (which is easy to check) is: $k \in \mathbb{Z} \iff -k \in \mathbb{Z}$

Proof of (i). Let $m, n \in \mathbb{Z}$. We distinguish 4 cases.

Case 1: both $m, n$ are positive. Then (i) follows from Problem 1.1.6(a).

Case 2: both $m, n$ are negative. Then $n' := -n$ and $m' = -m$ are positive integers, i.e. are natural numbers.

By Case 1, $m' + n' \in \mathbb{Z}$ and by the Remark above, $-(m' + n') \in \mathbb{Z}$. But $-(m' + n') = (-m') + (-n') = m + n$. So $m + n \in \mathbb{Z}$.

Case 3: one of them is zero, say $m = 0$. Then $m + n = n \in \mathbb{Z}$.

Case 4: one of them is positive and the other one is negative, say $m > 0$ and $n < 0$. Then $m \in \mathbb{N}$ and $n' := -n \in \mathbb{N}$.

Notice that $m + n = m - n'$.

If $m > n'$, then by Problem 1.1.8, $m - n' \in \mathbb{N}$. Since $\mathbb{N} \subseteq \mathbb{Z}$ and $m + n = m - n'$, we get $m + n \in \mathbb{Z}$.

If $m = n'$, then $m + n = 0 \in \mathbb{Z}$.

If $m < n'$, then $m + n = -(n' - m)$. We can apply again Problem 1.1.8 to get $n' - m \in \mathbb{N}$ Then $n' - m \in \mathbb{Z}$ and by the Remark above $-(n' - m) \in \mathbb{Z}$. Thus $m + n \in \mathbb{Z}$.

Proof of (ii). Write $m - n = m + (\neg n)$. If $m, n \in \mathbb{Z}$, then $m, -n \in \mathbb{Z}$ and by (i), it follows that $m - n \in \mathbb{Z}$.

Proof of (iii). Let $m, n \in \mathbb{Z}$. We distinguish 3 cases.

Case 1: $m$ is positive. Then $m \in \mathbb{N}$. We’ll prove that $m \cdot n \in \mathbb{Z}$ by induction on $m$. Fix $n \in \mathbb{Z}$ and consider the statement:

$$S(m) : m \cdot n \in \mathbb{Z}$$

The base case. $1 \cdot n = n \in \mathbb{Z}$, so $S(1)$ is true.

The induction step. Assume $S(k)$ is true, i.e. $k \cdot n \in \mathbb{Z}$. Then write $(k + 1) \cdot n = k \cdot n + n \in \mathbb{Z}$ (by (i)). This shows $S(k + 1)$ is true.

By the Principle of Mathematical Induction, $S(m)$ is true, for any $m \in \mathbb{N}$.

Case 2: $m = 0$. Then $m \cdot n = 0 \in \mathbb{Z}$.

Case 3: $m < 0$. Then $m' := -m \in \mathbb{N}$ and $m \cdot n = (m') \cdot n = m' \cdot (\neg n)$. In this manner, we reduced the problem to Case 1 with $m'$ instead of $m$ and $-n \in \mathbb{Z}$ instead of $n$.

Solution to Problem 1.1.11(a) Let $x \in \mathbb{Q}$ and $y \in \mathbb{R} \setminus \mathbb{Q}$. We are asked to show that $x + y \in \mathbb{R} \setminus \mathbb{Q}$.

Suppose for a contradiction that $x + y \in \mathbb{Q}$. Since $x \in \mathbb{Q}$, $-x \in \mathbb{Q}$ (we used here one of the field axioms for $\mathbb{Q}$). Then
\[ y = (x + y) + (-x) \in \mathbb{Q} \] (again, by one of the field axioms). But this contradicts \( y \in \mathbb{R} \setminus \mathbb{Q} \). Therefore, it must be that \( x + y \in \mathbb{R} \setminus \mathbb{Q} \).

**Remark 3.** Notice that Problem 1.1.10 asks to prove that \( \mathbb{Q} \) satisfies the field axioms (fact also mentioned during the lecture).

**(b)** Let \( x \in \mathbb{Q}, x \neq 0 \) and \( y \in \mathbb{R} \setminus \mathbb{Q} \) (this already implies that \( y \neq 0 \)). We are asked to show that \( x \cdot y \in \mathbb{R} \setminus \mathbb{Q} \).

Suppose for a contradiction that \( x \cdot y \in \mathbb{Q} \). Since \( x \in \mathbb{Q} \) and \( x \neq 0 \), \( x \) has a multiplicative inverse in \( \mathbb{Q} \). Then \( y = x^{-1} \cdot (x \cdot y) \in \mathbb{Q} \) (by one of the field axioms). But this contradicts \( y \in \mathbb{R} \setminus \mathbb{Q} \). Therefore, it must be that \( x \cdot y \in \mathbb{R} \setminus \mathbb{Q} \).

**Solution to Problem 1.1.13.** We have that \( S \subseteq \mathbb{R}, S \neq \emptyset \), \( S \) is bounded (so \( S \) has both infimum and supremum in \( \mathbb{R} \)). Denote \( \alpha := \inf(S) \) and \( \beta := \sup(S) \).

Since \( S \) is nonempty, \( S \) has at least one element, say \( x \). Since \( \alpha \) is a lower bound for \( S \), \( \alpha \leq x \). Since \( \beta \) is an upper bound for \( S \), \( \beta \geq x \).

Thus \( \alpha \leq x \) and \( x \leq \beta \). It follows that \( \alpha \leq \beta \).

**Remark 4.** Notice that if we are given \( \inf(S) = \sup(S) \), then \( \alpha = \beta \) which forces \( \alpha = x = \beta \). So if \( x \) denotes an arbitrary element of \( S \), we get that \( x \) must coincide with \( \alpha \). Hence \( \alpha \) is the only element in \( S \). This proves the result of **Problem 1.1.14.**

**Solutions to Problem 1.1.15.** ⇒ Suppose that \( S \) has a maximum \( M \). Since \( M \) is an upper bound of \( S \), so \( \sup S \leq M \). On the other hand, \( M \in S \) and \( \sup S \) is an upper bound of \( S \), so \( M \leq \sup S \). Therefore, we conclude that \( \sup S = M \in S \).

⇐ Suppose that \( S \) is bounded above and \( \sup S \in S \). We know that \( \sup S \) is an upper bound, and \( \sup S \in S \); These two conditions make \( \sup S \) a maximum of \( S \).

**Solutions to Problem 1.2.3.** Define \( a = \inf S \). Assume first that \( a \in S \). By the definition of the infimum, for any \( b \in S, a \leq b \). Therefore, \( a \) is the minimum of \( S \). Conversely, assume that \( m \) is the minimum of \( S \). Since \( m \in S \), so \( \inf S \leq m \). However, \( m \) is a lower bound of \( S \), so \( m \leq \inf S \). Therefore, we can conclude that \( m = \inf S \).

**Solution to Problem 1.2.5.** Suppose for a contradiction that \( a > 0 \). Then, by the Archimedean Property, there exists some \( n_0 \in \mathbb{N} \) s.t. \( \frac{1}{n_0} < a \). But this contradicts the hypothesis \( a \leq \frac{1}{n} \) for every \( n \in \mathbb{N} \).

2nd Method. From the hypothesis on \( a \), we have that \( a \) is a lower bound for the set \( S := \{ \frac{1}{n} : n \in \mathbb{N} \} \). From Ex. 1.2.4(a), \( \inf(S) = 0 \). Since \( \inf(S) \) is the greatest lower bound of \( S \), we deduce that \( a \leq 0 \).