Solution to Problem 2.1.12. Firstly, we show by induction that the sequence \( \{a_n\} \) given by

\[
a_{n+1} = \begin{cases} 
  a_n + \frac{1}{n} & \text{if } a_n^2 \leq 2 \\
  a_n - \frac{1}{n} & \text{if } a_n^2 > 2
\end{cases}, \quad a_1 = 1
\]

has positive terms. Before doing that let’s determine some of the first terms in this sequence: \( a_2 = a_1 + \frac{1}{1} = 2 \) (since \( a_1 < \sqrt{2} \)), \( a_3 = a_2 - \frac{1}{2} = \frac{3}{2} \) (since \( a_2 > \sqrt{2} \)), \( a_4 = a_3 - \frac{1}{3} = \frac{7}{6} < \sqrt{2} \). We see that \( \{a_n\} \) takes values both below and above \( \sqrt{2} \).

So, for each \( n \in \mathbb{N} \) consider the statement:

\[
S(n) : a_n > 0
\]

The base case is immediately verified for \( n = 1 \). For the induction step, assume \( S(n) \) is true, i.e. \( a_n > 0 \). In the case \( a_n^2 \leq 2 \) clearly \( a_n + \frac{1}{n} > 0 \). In the case \( a_n^2 > 2 \), suppose that \( a_n - \frac{1}{n} \leq 0 \), which implies \( a_n \leq \frac{1}{n} \leq 1 \) and consequently \( a_n^2 \leq 1 < 2 \) which contradicts \( a_n^2 > 2 \); it remains that \( a_n - \frac{1}{n} > 0 \). In any case, we obtained \( a_{n+1} > 0 \), i.e. \( S(n+1) \) is true. By the Principle of Induction, we deduce that \( S(n) \) is true for any \( n \in \mathbb{N} \).

Secondly, we prove

\[
P(n) : |a_n - \sqrt{2}| < \frac{2}{n} \iff -\frac{2}{n} < a_n - \sqrt{2} < \frac{2}{n}
\]

for every \( n \in \mathbb{N} \). Again, the base case is easily verified for \( n = 1 \). It is also easy to check \( n = 2 \). Next, assume \( P(n) \) is true (for some \( n \geq 2 \)). As above we’ll discuss the two cases:

Case 1. \( a_n^2 \leq 2 \) which is equivalent (as we now know that \( a_n > 0 \)) with \( a_n \leq \sqrt{2} \). Then \( a_{n+1} = a_n + \frac{1}{n} \), and adding \( \frac{1}{n} \) to the inequalities given by \( P(n) \), we obtain:

\[
-\frac{1}{n} < \frac{a_n + \frac{1}{n}}{n} - \sqrt{2} < \frac{3}{n} \iff -\frac{1}{n} < a_{n+1} - \sqrt{2} < \frac{3}{n}
\]

Out of this, we can use the left inequality (the right one is too weak) and since \( -\frac{2}{n+1} \leq -\frac{1}{n} \) for any \( n \in \mathbb{N} \), we deduce

\[
-\frac{2}{n+1} < a_{n+1} - \sqrt{2}
\]

To get the right inequality of \( P(n+1) \), we use \( a_n \leq \sqrt{2} \):

\[
a_n + \frac{1}{n} \leq \sqrt{2} + \frac{1}{n} \Rightarrow a_{n+1} - \sqrt{2} \leq \frac{1}{n} < \frac{2}{n+1} \quad \text{(here, we needed } n \geq 2)\]

Therefore

\[
-\frac{2}{n+1} < a_{n+1} - \sqrt{2} < \frac{2}{n+1},
\]

i.e. \( P(n+1) \) is true.

Case 2. \( a_n^2 > 2 \) which is equivalent (since \( a_n > 0 \)) with \( a_n > \sqrt{2} \). We mirror the arguments from Case 1. Here, \( a_{n+1} = a_n - \frac{1}{n} \) so we add \(-\frac{1}{n}\) to the inequalities given by \( P(n) \):

\[
\frac{3}{n} < \frac{a_n - \frac{1}{n}}{n} - \sqrt{2} < \frac{1}{n} \iff \frac{3}{n} < a_{n+1} - \sqrt{2} < \frac{1}{n}
\]

Out of this, we can use the right inequality (the left one is too weak) and since \( \frac{1}{n} \leq \frac{2}{n+1} \) for any \( n \in \mathbb{N} \), we deduce

\[
a_{n+1} - \sqrt{2} \leq \frac{2}{n+1}
\]

To get the left inequality of \( P(n+1) \), we use \( a_n > \sqrt{2} \):

\[
a_n - \frac{1}{n} > \sqrt{2} - \frac{1}{n} \Rightarrow a_{n+1} - \sqrt{2} > -\frac{1}{n} \geq -\frac{2}{n+1}
\]

Therefore

\[
-\frac{2}{n+1} < a_{n+1} - \sqrt{2} < \frac{2}{n+1},
\]
i.e. \( P(n+1) \) is true.

So, in both cases we derived \( P(n+1) \), provided \( P(n) \). By the Induction Principle, \( P(n) \) is true for any \( n \geq 2 \), and since it was explicitly checked for \( n = 1 \), we conclude that \( P(n) \) is true for any \( n \in \mathbb{N} \).

**Solution to Problem 2.1.14.** Let’s start by computing some terms:

\[
egin{align*}
s_1 &= \frac{1}{2}, \quad s_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}, \quad s_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}, \quad \ldots
\end{align*}
\]

We notice a pattern here, namely \( s_n = \frac{n}{n+1} \). We try to prove this formally by induction. But first, notice that we have the following recursion formula:

\[
s_{n+1} = s_n + \frac{1}{(n+2)(n+1)}
\]

So for each \( n \in \mathbb{N} \) consider:

\[ P(n) : s_n = \frac{n}{n+1} \]

The base case is immediately checked since we computed the first term. Next, assume \( P(n) \) is true. Then, using (1)

\[
s_{n+1} = s_n + \frac{1}{(n+2)(n+1)} = \frac{n^2 + 2n + 1}{(n+2)(n+1)} = \frac{n+1}{n+2}
\]

Hence, \( P(n+1) \) is true.

By the Induction Principle \( P(n) \) is true for all \( n \in \mathbb{N} \), i.e.

\[
s_n = \frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}}
\]

which goes to \( \frac{1}{1+0} = 1 \) as \( n \to \infty \).

2nd Method. The generic term \( \frac{1}{(k+1)k} \) can be written as

\[
\frac{1}{(k+1)k} = \frac{1}{k} - \frac{1}{k+1}
\]

Then

\[
s_n = \sum_{k=1}^{n} \frac{1}{(k+1)k} = \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \ldots + \left( \frac{1}{n} - \frac{1}{n+1} \right)
\]

\[
= 1 - \frac{1}{n+1}
\]

and consequently

\[
\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( 1 - \frac{1}{n+1} \right) = 1 - \lim_{n \to \infty} \frac{1}{n+1} = 1
\]

**Solution to Problem 2.1.15.**

a. Let \( c > 0 \) be an arbitrary real number. By the Archimedean property, there exists \( N_1 \in \mathbb{N} \) such that \( N_1 > c \). Let \( N = \max\{N_1, 21\} \). Thus, \( N > c \) and \( N \geq 21 \). Consequently, for all \( n \geq N \),

\[
n^3 - 4n^2 - 100n > n^3 - 15n^2 + 100n = n(n-20)(n+5) \geq N(N-20)(N+5) > c,
\]

for all \( n \geq N \).

As \( c \) is arbitrary, we conclude that \( \lim_{n \to \infty} n^3 - 4n^2 - 100n = \infty \).

b. Let \( c > 0 \) be arbitrary real number. By the Archimedean property, there exists \( N \in \mathbb{N} \) such that \( N > c^2 \). Hence, for all \( n \geq N \),

\[
\sqrt{n} - \frac{1}{n^2} + 4 \geq \sqrt{n} - 1 + 4 > \sqrt{n} \geq \sqrt{N} > c.
\]

As \( c \) is arbitrary, we conclude that \( \lim_{n \to \infty} \sqrt{n} - \frac{1}{n^2} + 4 = \infty \).

**Solution to Problem 2.1.16.** Write

\[
\sqrt{n+1} - \sqrt{n} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}
\]
Now, notice that
\[
0 \leq \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{\sqrt{n} + \sqrt{n}} = \frac{1}{2\sqrt{n}}, \forall n \in \mathbb{N}
\]
Since \(\lim_{n \to \infty} \frac{1}{2\sqrt{n}} = 0\), by the Squeeze Theorem, we get
\[
\lim_{n \to \infty} (\sqrt{n+1} - \sqrt{n}) = 0
\]
For (b), we have
\[
(\sqrt{n+1} - \sqrt{n})\sqrt{n} = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + 1} = \frac{1}{\sqrt{1 + \frac{1}{n}} + 1}
\]
whose denominator clearly converges to \(\sqrt{1} + 1\), hence
\[
\lim_{n \to \infty} (\sqrt{n+1} - \sqrt{n})\sqrt{n} = \frac{1}{2}
\]
**Remark.** We cannot use the Squeeze Theorem as in (a) with the bounds
\[
0 \leq (\sqrt{n+1} - \sqrt{n})\sqrt{n} \leq \left(\frac{1}{2\sqrt{n}}\right)\sqrt{n}
\]
because the last inequality is too weak.

For (c), we use the computation from (b):
\[
(\sqrt{n+1} - \sqrt{n})n = (\sqrt{n+1} - \sqrt{n})\sqrt{n} = \frac{\sqrt{n}}{\sqrt{1 + \frac{1}{n}} + 1}
\]
which holds for every \(n \in \mathbb{N}\). Since the numerator goes to \(+\infty\) and the denominator converges to 2, we get
\[
\lim_{n \to \infty} (\sqrt{n+1} - \sqrt{n})n = +\infty
\]

**Solution to Problem 2.1.17.** We start by writing the \(\varepsilon\) – \(N\) definitions for the two limits:
\[
(*) \lim_{n \to \infty} a_n = +\infty : \forall c > 0, \exists N = N(c) \in \mathbb{N} \text{ s.t. } a_n > c, \forall n \geq N;
\]
\[
(**) \lim_{n \to \infty} \frac{1}{a_n} = 0 : \forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N} \text{ s.t. } \left|\frac{1}{a_n} - 0\right| < \varepsilon, \forall n \geq N.
\]
Since \(a_n > 0\), we have:
\[
(3) \quad \left|\frac{1}{a_n} - 0\right| < \varepsilon \iff \frac{1}{a_n} < \varepsilon \iff a_n > \frac{1}{\varepsilon}
\]
We were asked to show (*) \iff (**).

\(\Rightarrow\): Assume (*) is true. In order to prove (**), let \(\varepsilon > 0\). Set \(c := \frac{1}{\varepsilon}\) and apply (*).
So, there is \(N \in \mathbb{N}\) s.t. \(a_n > c \iff a_n > \frac{1}{\varepsilon}\) for all \(n \geq N\). By the equivalence (3), we get \(\left|\frac{1}{a_n} - 0\right| < \varepsilon\) for all \(n \in \mathbb{N}\).

\(\Leftarrow\): Assume (**) is true. In order to prove (*), let \(c > 0\). Set \(\varepsilon := \frac{1}{c}\) and apply (**).
So, there is \(N \in \mathbb{N}\) s.t. \(\left|\frac{1}{a_n} - 0\right| < \varepsilon \iff a_n > \frac{1}{\varepsilon} = c\) for all \(n \geq N\) (here we used again (3)). Therefore, we have \(a_n > c\) for all \(n \in \mathbb{N}\).
Solution to Problem 2.2.1. (a) false: \((-1)^n\) is bounded but is not convergent.

(b) false: \(\left\{ \frac{1}{n} \right\}\) is a sequence of positive numbers with limit 0.

(c) false: \(n^2 + 1 > n\) and \(n \to \infty\).

(d) false: by invoking the sequential density of the rationals for \(\sqrt{2}\), we infer that there exists a sequence of rational numbers converging to \(\sqrt{2}\) which is irrational.

An explicit sequence (you’ll see it again later on) is \(\{a_n\}\) with \(a_n = \left(1 + \frac{1}{n}\right)^n \in \mathbb{Q}\) and \(\lim_{n \to \infty} a_n = e \in \mathbb{R} \setminus \mathbb{Q}\).

(e) false: the sequence \(\left\{ \frac{1}{n} \right\}\) is in \((0, 2)\) but its limit is not.

Solution to Problem 2.2.2. Denote \(S := (-\infty, 0] = \{x \in \mathbb{R} : x \leq 0\}\). Let \(\{x_n\}\) be a convergent sequence in \(S\) (i.e. its terms are all in \(S\)). Say \(x = \lim_{n \to \infty} x_n\). We need to show that \(x \in S\).

Since \(x_n \in S\), we have \(x_n \leq 0\) for every \(n \in \mathbb{N}\). Then \(-x_n \geq 0\) for every \(n \in \mathbb{N}\) and \(\lim_{n \to \infty} (-x_n) = -x\). By Lemma 2.21, we get \(-x \geq 0\), equivalently \(x \leq 0\). Hence, \(x \in S\).

**Suggestion:** Try to go directly through the analogous proof to Lemma 2.21 to solve this, i.e. try a proof by contradiction. I’ve did something similar for Problem 2.1.6.

Solution to Problem 2.2.3. Let \(a \in \mathbb{R}\). Since \(\mathbb{Q}\) is sequentially dense in \(\mathbb{R}\), there exists \(\{a_n\}\) a sequence in \(\mathbb{Q}\) such that \(a_n \to a\), i.e. \(\lim_{n \to \infty} a_n = a\). If we could find irrational terms \(b_n\) such that \(b_n \to 0\), then \(\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n = a\).

So, the problem is reduced to finding a sequence \(\{b_n\}\) of irrational numbers that converges to 0. Take \(b_n = \frac{1}{n+\pi}\).

Is easy to check that \(0 < b_n < \frac{1}{n}\) and so \(b_n \to 0\). We know that \(\pi\) is irrational. If we suppose \(\frac{1}{n+\pi}\) is rational, then \(n+\pi\) is rational, and further \(\pi = (n+\pi) - n\) is rational. Therefore, \(\{b_n\}\) is a sequence of irrational numbers. Which satisfies our requirements.

**Remark.** You can imagine other choices for \(\{b_n\}\). For example, \(\left\{ \frac{\pi}{n} \right\}, \left\{ \frac{1}{\sqrt{n}} \right\} \), etc. will work equally well. More sophisticated attempts would be \(\left\{ \frac{1}{\sqrt{n}} \right\}\) which almost satisfies this requirement; it merely fails because some of its terms are rationals (when \(n\) is a perfect square, \(1/\sqrt{n}\) is rational). You could try to fix this by choosing \(b_n = \frac{1}{\sqrt{n}\sqrt{n+1}}\), or by excluding the rational terms and reindex the elements you’re left with.

Solution to Problem 2.2.4. Take the sequence \(\{b_n\}\) from the previous problem, e.g. \(b_n = \frac{1}{n+\pi}\).

Then \(b_n \in \mathbb{R} \setminus \mathbb{Q}\), but \(\lim_{n \to \infty} b_n = 0 \notin \mathbb{R} \setminus \mathbb{Q}\).

Solution to Problem 2.3.3. “⇒”: Assume \(\{a_n\}\) converges to some \(a \in \mathbb{R}\). By the Product Theorem, \(a_n \cdot a_n\) goes to \(a \cdot a = a\) as \(n \to \infty\), thus \(\{a_n\}^2\) converges to \(a^2\).

“⇐”: Assume \(\{a_n\}\) converges. In particular, \(\{a_n^2\}\) is bounded, so there exists \(M > 0\) s.t. \(|a_n^2| \leq M\). Then

\[|a_n| = \sqrt{a_n^2} \leq \sqrt{M}, \forall n \in \mathbb{N}\]

It follows that \(\{a_n\}\) is bounded. But from the hypothesis, we know that \(\{a_n\}\) is monotone. By MCT, \(\{a_n\}\) converges.

**Remark:** Sometimes, it’s useful to think about monotonicity via the following simple facts:

- \(\{a_n\}\) is monotonically increasing if and only if \(a_{n+1} - a_n\) is zero or positive for every \(n \in \mathbb{N}\);
- \(\{a_n\}\) is monotonically decreasing if and only if \(a_{n+1} - a_n\) is zero or negative for every \(n \in \mathbb{N}\).

Solution to Problem 2.3.4. By the hypothesis, \(|a| < 1\) and for every \(\varepsilon > 0\) there exists \(N_\varepsilon \in \mathbb{N}\) s.t.

\[|a_n - a| < \varepsilon, \forall n \geq N_\varepsilon\]

We’ll use the Comparison Lemma, so we need to place an upper bound on \(|a_n^m|\).

Since \(|a_n - a| \geq |a_n| - |a| \iff |a_n| \leq |a| + |a_n - a|\), we also have

\[|a_n^m| = |a_n|^m < (|a| + \varepsilon)^m, \forall n \in \mathbb{N}\]
Now we know that $b^n$ converges to 0 provided that $|b| < 1$. So we need to make a choice of $\varepsilon$ that assures $|a| + \varepsilon < 1$. For example we can take $\varepsilon_0 := \frac{1 - |a|}{2}$. We definitely have $\varepsilon_0 > 0$ and also $|a| + \varepsilon_0 = \frac{|a| + 1}{2} < \frac{1 + 1}{2} = 1$. So, there exist $b := |a| + \varepsilon_0$ and some index $N_{\varepsilon_0} \in \mathbb{N}$ s.t. $|a^n| < b^n$ for all $n \geq N_{\varepsilon_0}$. Since $\lim_{n \to \infty} b^n = 0$, it follows that $\lim_{n \to \infty} a^n = 0$.

**Solution to Problem 2.3.5.** Let $d = \frac{1 - |c|}{|c|} > 0$, then $c = \frac{1}{1 + d}$. The Binomial Formula yields

$$|c|^n = \left(\frac{1}{1 + d}\right)^n = \frac{1}{1 + nd + \ldots} < \frac{1}{dn}.$$  

**Solution to Problem 2.3.6. a.** From Problem 2.3.5, we have

$$|c^n - 0| < \left|\frac{1}{dn} - 0\right| \quad \text{for all } n \geq 1.$$  

Since $\lim_{n \to \infty} \frac{1}{dn} = 0$, so $\lim_{n \to \infty} c^n = 0$ by the Comparison Lemma.

b. Multiplying both sides of the inequality in a. by $\sqrt{n}$, we obtain

$$|\sqrt{n}c^n - 0| < \left|\frac{1}{d\sqrt{n}} - 0\right| \quad \text{for all } n \geq 1.$$  

Since $\lim_{n \to \infty} \frac{1}{d\sqrt{n}} = 0$, so $\lim_{n \to \infty} \sqrt{n}c^n = 0$ by the Comparison Lemma.

**Solution to Problem 2.3.8.** First, notice that $\{s_n\}$ is monotonically increasing:

$$s_{n+1} = b_1r + b_2r^2 + \ldots + b_nr^n + b_{n+1}r^{n+1} = s_n + b_{n+1}r^{n+1} \geq s_n, \quad \forall n \in \mathbb{N}.$$  

Also, it’s easy to see that $s_n \geq 0$. Then since $\{b_n\}$ is a bounded sequence in $[0, \infty)$, there exists $M > 0$ s.t. $b_n \leq M$ for all $n \in \mathbb{N}$.

$$|s_n| = s_n = b_1r + b_2r^2 + \ldots + b_nr^n \leq Mr + Mr^2 + \ldots + Mr^n = Mr(1 + r + \ldots + r^{n-1}) = Mr \frac{1 - r^n}{1 - r}.$$  

Since

$$\frac{1 - r^n}{1 - r} \leq \frac{1}{1 - r}, \quad \forall n \in \mathbb{N}.$$  

there exists $B := Mr \frac{1}{1 - r} > 0$ s.t.

$$s_n \leq B, \quad \forall n \in \mathbb{N},$$  

i.e. $\{s_n\}$ is bounded above. Since it is also bounded below (e.g. by 0), $\{s_n\}$ is bounded. By MCT, $\{s_n\}$ converges.

**Remark:** Notice that we don’t have an explicit formula for $\lim_{n \to \infty} s_n$, we just know it exists (and it is a real number).