HW 6 PROBLEMS

From Rudin Chapter 3 solve: 4,5,6,7,8,14(a,b,c,d)

Also solve the following problems:

Problem 1. Let l^2 denote the space of sequences $\{x_n\}$ of real numbers such that the series

$$\sum_{n=1}^{\infty} x_n^2$$

converges. The norm of such a sequence is defined by

$$\|\{x_n\}\| = \left(\sum_{n=1}^{\infty} x_n^2\right)^{\frac{1}{2}}$$

For two sequences $\{x_n\}$ and $\{y_n\}$ we define

$$d(\{x_n\}, \{y_n\}) = \|\{x_n - y_n\}\|$$

(i) Show that the distance is well-defined and that l^2 with this distance is a metric space. You may want to remember Cauchy-Schwarz.

(ii) For each $m \geq 1$, consider the sequence s_m whose terms are all equal to 0 except for the m'th term which is 1. That is,

$$s_m = (0, 0, .., 0, 1, 0, ...).$$

Show that for each $m \geq 1$, s_m is an element in l^2 and show that the resulting sequence $\{s_m\}$ (sequence in l^2) is not Cauchy in l^2.

(iii) In the metric space l^2 consider the closed unit ball of center the zero sequence i.e. the set

$$K = \{\{x_n\} \in l^2 : \|\{x_n\}\| \leq 1\}.$$

Show that K is closed and bounded but not compact by exhibiting a sequence in K which does not have a convergent subsequence.

Problem 2. Assume that the series $\sum_{n=1}^{\infty} a_n^2$ converges; prove that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

Problem 3. Let $\{a_n\}$ be a monotonically decreasing sequence of positive numbers with the property that $10a_{2n} \leq a_n, \forall n \in \mathbb{N}$. Prove that $\sum_{n=1}^{\infty} a_n$ converges.