HOMEWORK 1 SOLUTIONS

Ex 1.1 We argue by contradiction: Suppose $r + x_1$ is a rational number. Since $\mathbb{Q}$ is a field and $r \in \mathbb{Q}$, $-r \in \mathbb{Q}$. Therefore $x_1 = (r + x_1) + (-r) \in \mathbb{Q}$ which contradict the hypothesis. Therefore get $r + x_1$ is not rational.
Suppose rx_2 is a rational number. Since $\mathbb{Q}$ is a field and $r \in \mathbb{Q}$ $(r \neq 0)$, $1/r \in \mathbb{Q}$. Therefore$x_2 = (rx_2)(1/r) \in \mathbb{Q}$, which contradict the hypothesis.. Thus rx_2 is irrational.

Ex 1.2 We use contradiction. Assume there is a rational number q whose square is 12, then r can be represented as n/m, where $n, m \in \mathbb{Z}$ and their greatest common divisor is 1 (denoted bygcd$(m, n) = 1$). Then $r^2 = (n/m)^2 = 12$, which is equivalent to $n^2 = 12m^2$. Note that 3 can devide the right hand side $12m^2$, thus 3 can devide the left hand side n^2 (denoted by $3|n^2$). Since 3 is prime, $3|n$, thus n can be represented as $3n_1$ $(n_1 \in \mathbb{Z})$. Then we get $9n_1^2 = 12m^2$, which is equivalent to $3n_1^2 = 4m^2$. Notice now on the left hand $3|3n_1^2$, we get $3|4m^2$. Since 3 is prime, $3|m$. Now we get $3|n$ and $3|m$, which contradict the assumption that $\gcd(m.n) = 1$. Thus such q does NOT exisit.

Ex1.5 First notice A is nonempty and bounded below. Thus $\inf A$ exists by THM 1.19 in the textbook. Since A is nonempty and bounded below, $-A = \{-x \mid x \in A\}$ is nonempty snd bounded above. Therefore $\sup(-A)$ exists. Let $\alpha = \inf A$ and $\beta = -\sup(-A)$. To show $\alpha = \beta$, we only need to show $\alpha \leq \beta$ and $\alpha \geq \beta$.

By definition of α, α is a lower bound of A, i.e. $\alpha \leq x$, $\forall\, x \in A$. By multiplying -1 on both sides, we get $-\alpha \geq -x$, $\forall\, x \in A$, which is equivalent to $-\alpha$ is an upper bound of $-A$. Thus $-\alpha \geq -\beta$ by the definition of β; from this it follows that $\alpha \leq \beta$.

On the other hand, by definition of β, $-\beta$ is an upper bound of $-A$, i.e. $-\beta \geq -x$, $\forall\, x \in A$. That is equivalent to $\beta \leq x$, $\forall\, x \in A$. This means β is a lower bound of A. Then by the definition of α, $\alpha \geq \beta$.

Combining the two inequalities, we obtain $\alpha = \beta$ as desired.