HOMEWORK 1 SOLUTIONS

Ex 1.1 We argue by contradiction: Suppose \( r + x_1 \) is a rational number. Since \( \mathbb{Q} \) is a field and \( r \in \mathbb{Q} \), \( -r \in \mathbb{Q} \). Therefore \( x_1 = (r + x_1) + (-r) \in \mathbb{Q} \) which contradict the hypothesis. Therefore get \( r + x_1 \) is not rational.

Suppose \( rx_2 \) is a rational number. Since \( \mathbb{Q} \) is a field and \( r \in \mathbb{Q} \) \((r \neq 0)\), \( 1/r \in \mathbb{Q} \). Therefore \( x_2 = (rx_2)(1/r) \in \mathbb{Q} \), which contradict the hypothesis. Thus \( rx_2 \) is irrational.

Ex 1.2 We use contradiction. Assume there is a rational number \( q \) whose square is 12, then \( r \) can be represented as \( n/m \), where \( n, m \in \mathbb{Z} \) and their greatest common divisor is 1 (denoted by \( \gcd(m,n) = 1 \)). Then \( r^2 = (n/m)^2 = 12 \), which is equivalent to \( n^2 = 12m^2 \). Note that 3 can divide the right hand side \( 12m^2 \), thus 3 can divide the left hand side \( n^2 \) (denoted by \( 3|n^2 \)). Since 3 is prime, \( 3|n \), thus \( n \) can be represented as \( 3n_1 \) \((n_1 \in \mathbb{Z})\). Then we get \( 9n_1^2 = 12m^2 \), which is equivalent to \( n_1^2 = 4m^2 \). Notice now on the left hand \( 3|n_1 \), we get \( 3|m \). Now we get \( 3|n \) and \( 3|m \), which contradict the assumption that \( \gcd(m,n) = 1 \). Thus such \( q \) does NOT exist.

Ex 1.5 First notice \( A \) is nonempty and bounded below. Thus \( \inf A \) exists by THM 1.19 in the textbook. Since \( A \) is nonempty and bounded below, \( -A = \{-x \mid x \in A\} \) is nonempty and bounded above. Therefore \( \sup(-A) \) exists. Let \( \alpha = \inf A \) and \( \beta = -\sup(-A) \). To show \( \alpha = \beta \), we only need to show \( \alpha \leq \beta \) and \( \alpha \geq \beta \).

By definition of \( \alpha \), \( \alpha \) is a lower bound of \( A \), i.e. \( \alpha \leq x \), \( \forall x \in A \). By multiplying \(-1\) on both sides, we get \( -\alpha \geq -x \), \( \forall x \in A \), which is equivalent to \( -\alpha \) is an upper bound of \(-A\). Thus \( -\alpha \geq -\beta \) by the definition of \( \beta \); from this it follows that \( \alpha \leq \beta \).

On the other hand, by definition of \( \beta \), \( -\beta \) is an upper bound of \(-A\), i.e. \( -\beta \geq -x \), \( \forall x \in A \). That is equivalent to \( \beta \leq x \), \( \forall x \in A \). This means \( \beta \) is a lower bound of \( A \). Then by the definition of \( \alpha \), \( \alpha \geq \beta \).

Combining the two inequalities, we obtain \( \alpha = \beta \) as desired.