Exercise 1.6

(a) Let $s = mq = np$. By Theorem 1.21, there is only one $c > 0$ such that $c^{nq} = b^s$. Then notice $(b^m)^{1/n}$ and $(b^p)^{1/q}$ both have this property.

(b) Suppose $r = \frac{p}{q}$, $s = \frac{m}{n}$. Then compute b^{r+s} using the law of exponents for integers. By Corollary of Thm 1.21, the result follows from part (a).

(c) A generic element in $B(r)$ is of the form b^t with $t \le r$, thus $b^t \le b^r$. This makes b^r the maximum element in $B(r)$, therefore $b^r = supB(r)$.

Here we are using the fact that if $p < q, p, q \in \mathbb{Q}$ and $b > 1$, then $b^p < b^q$. Since $b > 1$ we claim that for any $m, n \in Z$, the following holds true:

$$m < n \Leftrightarrow b^m < b^n$$

Indeed, $b^n - b^m = b^m(b^{n-m} - 1) > 0 \Leftrightarrow b^{n-m} - 1 > 0 \Leftrightarrow b^{n-m} > 1 \Leftrightarrow n - m > 0$.

Getting back to our claim, let $p = \frac{m_1}{n_1}, q = \frac{m_2}{n_2}$ with $m_1, m_2, n_1, n_2 \in \mathbb{Z}, n_1, n_2 > 0$. $p < q$ reads $m_1 n_2 < m_2 n_1$ and from our argument earlier we obtain $b^{m_1 n_2} < b^{m_2 n_1}$. We write $b^{m_1 n_2} = (b^p)^{n_1 n_2}, b^{m_2 n_1} = (b^q)^{n_1 n_2}$, let $n = n_1 n_2$ and continue with

$$0 < b^{m_2 n_1} - b^{m_1 n_2} = (b^q)^{n_1 n_2} - (b^p)^{n_1 n_2} = (b^q)^n - (b^p)^n = (b^q - b^p)\left((b^q)^{n-1} + (b^q)^{n-2} b^p + ... + (b^p)^{n-1}\right)$$

Since the term in the parentheses is positive, we conclude with $b^q - b^p > 0$, thus $b^q > b^p$ and we are done.

(d) To have a more elegant solution we remark that if we define $\bar{B}(x) = \{b^q : q < x\}$ and note that for $x \notin \mathbb{Q}$, $\bar{B}(x) = B(x)$ while for $x \in \mathbb{Q}$, $\bar{B}(x) = B(x) \setminus \{b^x\}$. Thus if $x \notin \mathbb{Q}$, $\sup B(x) = \sup \bar{B}(x)$. If $x \in Q$, then b^x is an upper bound for $\bar{B}(x)$ and we need to show it is the least upper bound. We compute $b^x - b^{x-\frac{1}{n}} = b^{x-\frac{1}{n}}(b^{\frac{1}{n}} - 1) = b^{x-\frac{1}{n}} \frac{b-1}{b^{\frac{n-1}{n}} + b^{\frac{n-2}{n}} + ... + 1} \le \frac{1}{n} b^x (b-1)$ and the last expression can be made as small as we want by the Archimedian property, therefore $b^x = \sup \bar{B}(x)$.

Given two subsets X, Y of $\mathbb{R}$, we define $X \cdot Y = \{x \cdot y | x \in X, y \in Y\}$. Notice that $B(x) \cdot B(y) = \{b^r b^s : r < x, s < y\} = \{b^{r+s} : r < x, s < y\}$. It is obvious that $r + s < x + y$, therefore $B(x) \cdot B(y) \subset B(x+y)$. It takes a little more work to show the reverse $B(x+y) \subset B(x) \cdot B(y)$. Given an element in $B(x+y)$, we know it has the form b^q with $q \in \mathbb{Q}$ and $q < x + y$. Our goal is to show that we can find $r, s \in \mathbb{Q}$ with $q = r + s, r < x, s < y$ since then $b^q = b^r \cdot b^s \in B(x) \cdot B(y)$ and we are done. Let $\epsilon = (x + y) - q > 0$. We know that there exist $r \in Q$ with $x - \epsilon < r < x$ and we let $s = q - r$. We have $s < q - (x - \epsilon) = y$ and $s \in \mathbb{Q}$ (since $q, r \in \mathbb{Q}$). We conclude with $B(x+y) = B(x) \cdot B(y)$.

The argument is complete if we prove that $\sup(X \cdot Y) = \sup X \cdot \sup Y$ for $X, Y \subset \mathbb{R}_+$. If $\sup X = 0$ or $\sup Y = 0$, then $X = \{0\}$ or $Y = \{0\}$ and the conclusion is trivial. Therefore, in what follows, we have that $\sup X > 0$ and $\sup Y > 0$. For any $x \in X$ and any $y \in Y$ we have $x \le \sup X, y \le \sup Y$, therefore $xy \le \sup X \sup Y, \forall x \in X, y \in Y$. This implies that $\sup X \sup Y$ is an upper bound for $X \cdot Y$, thus $\sup(X \cdot Y) \le \sup X \sup Y$.

Next we have that $xy \le \sup(X \cdot Y), \forall x \in X, y \in Y$. We fix $x \in X$ with $x > 0$ and conclude that $y \le \frac{\sup(X \cdot Y)}{x}, \forall y \in Y$, thus $\frac{\sup(X \cdot Y)}{x}$ is an upper bound for Y and $\sup Y \le \frac{\sup(X \cdot Y)}{x}$. From this it follows that $x \sup Y \le \sup(X \cdot Y)$ for any $x \in X$ with $x > 0$; but now it is obvious that it holds for $x = 0$ as well. Since $supY > 0$, we obtain that $x \le \frac{\sup(X \cdot Y)}{\sup Y}$ for all $x \in X$ hence $\sup X \le \frac{\sup(X \cdot Y)}{\sup Y}$ and further that $\sup X \sup Y \le \sup(X \cdot Y)$.

Since we established the two inequalities $\sup(X \cdot Y) \leq \sup X \sup Y$ and $\sup X \sup Y \leq \sup(X \cdot Y)$, we can conclude $\sup X \sup Y = \sup(X \cdot Y)$. This finishes our argument.

Exercise 1.8

By part (a) of Proposition 1.18, either i of $-i$ is positive. So $-1 = i^2 = (-i)^2 = -1$ has to be positive. Since $(-1)^2 = 1$ is positive, we have both -1 and 1 are positive. Contradition with Proposition 1.18 part (a) for an ordered field.

Exercise 1.10 Compute following the rule of complex numbers and notice when taking off the square root, you need to consider the sign.

Exercise 1.11 If $z = 0$, we can let $r = 0$ and $w = 1$ but in this case w is not unique. Otherwise, let $r = |z|$ and $w = \frac{z}{|z|}$. Now, w and r are uniquely determined by z. This is because $z = rw$ implies $r = r|w| = |rw| = |z|$.

Exercise 1.12

By Theorem 1.33, the claim has been shown when $n = 2$ in part (e). We can apply the result and do induction on n. Assume it holds for $n-1$ and show the result for n.

$$\begin{aligned}
|z_1 + z_2 + \cdots + z_n| &= |(z_1 + \cdots + z_{n-1}) + z_n| && (1)\\
&\leq |z_1 + z_2 + \cdots + z_{n-1}| + |z_n| && (2)\\
&\leq |z_1| + |z_2| + \cdots + |z_{n-1}| + |z_n|. && (3)
\end{aligned}$$

Exercise 1.13

Since $x = x - y + y$, apply the triangle inequality and we have

$$|x| \leq |x-y| + |y|.$$

Similarly, $y = y - x + x$ gives

$$|y| \leq |y-x| + |x|.$$

Combining them gives $\big||x| - |y|\big| \leq |x-y|$.

Exercise 1.16 You can interpret geometrically first to get a sense of the problem. (a) Assume $\mathbf{w}$ is a vector satisfying

$$\mathbf{w} \cdot (\mathbf{x} - \mathbf{y}) = 0$$
$$|\mathbf{w}|^2 = r^2 - \frac{d^2}{4}.$$

From linear algebra, we know only one component of a solution $\mathbf{w}$ is fixed and the other components of the solution is arbitrary. In addition, if $\mathbf{w} \neq \mathbf{0}$, then there is a unique positive number $s > 0$ such that $s\mathbf{w}$ satisfies both equations given $r^2 - \frac{d^2}{4} > 0$. So we have inifitely many solutions.

(b) Notice with the condition $2r = d$, we have

$$|\mathbf{x} - \mathbf{y}| = d = |\mathbf{x} - \mathbf{z}| + |\mathbf{z} - \mathbf{y}|.$$

The proof of the triangle inequality shows that the equality can hold only when the two vectors is a scalar multiple of the other. Here $\mathbf{x} - \mathbf{z} = s(\mathbf{z} - \mathbf{y})$ for some $s > 0$. The assumption gives $s = 1$ immediately then. Thus $\mathbf{z}$ is uniquely determined as

$$\mathbf{z} = \frac{\mathbf{x} + \mathbf{y}}{2}.$$

(c) Now if $2r < d$,

$$|\mathbf{x} - \mathbf{y}| = d > |\mathbf{x} - \mathbf{z}| + |\mathbf{z} - \mathbf{y}|.$$

Contradiction with the triangle inequality. So no solutions for $\mathbf{z}$.

Note: When $k = 2$, there are exactly 2 solutions in case (a). When $k = 1$, there is no solution in case (a). The other results in part (b) and (c) does not need to be modified.

Exercise 1.17

Compute $|\mathbf{x} + \mathbf{y}|^2 = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y})$ and $|\mathbf{x} - \mathbf{y}|^2 = (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})$ and expand them. Sum up and you will have the right hand side of the equation. Geometric interpretation: the sum of the squares on the diagonals of a parallelogram equals the sum of the squares on the sides.

Exercise 1.18

If $\mathbf{x}$ has any component equal to 0, then $\mathbf{y}$ can be taken to be 1 on the corresponding component and all other components to be 0. If all components of $\mathbf{x}$ is nonzero, we can let $\mathbf{y} = (-x_2, x_1, 0, \ldots, 0)$. It is not true for $k = 1$ though since the product of two nonzero real numbers is nonzero.