Solution to Homework 3

Haiyu Huang

October 17, 2018

Remark. *In homework 1 problem 1.2 that is to show* $\sqrt{12}$ *is irrational, many falsely asserted that* $12 \mid m^2$ *implies* $12 \mid m$*. While for any* p *prime* $p \mid m^2$ *implies* $p \mid m$*, this is not true in general for composite number (take* $m = 6$ *in this case). What is true is that for any prime* p*,* $p \mid ab \implies p \mid a$ *or* $p \mid b$*. This is called Euclid's lemma and can be taken as the defining property of prime numbers.*

2.2

A complex number z is said to be algebraic if there are integers $a_0, \cdots, a_n$, not all zero such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove the set of algebraic numbers is countable.

Proof. Let A_N be the set of algebraic numbers satisfying an equation as above with $n + |a_0| + |a_1| + \cdots + |a_n| = N$. A_N is finite because there are finitely many equations satisfying this condition and each equation has finitely many solution (fundamental theorem of algebra). So the set of all algebraic numbers, $\bigcup_{N=2}^{\infty} A_N$, is at most countable. Since all integers are algebraic, the set of algebraic numbers is countable. ■

2.4

Is the set of irrational real numbers countable?

Proof. If $\mathbb{R} \setminus \mathbb{Q}$ were countable, then $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ would be countable, a contradiction. ■

2.7

Finite union of closures is closure of finite union. Not true in general for infinite union.

Proof. (a) $A_1 \subset A_1 \cup A_2$ so $\overline{A_1} \subset \overline{A_1 \cup A_2}$ (Verify!). By symmetry $\overline{A_2} \subset \overline{A_1 \cup A_2}$. Hence $\overline{A_1} \cup \overline{A_2} \subset \overline{A_1 \cup A_2}$. Conversely, $A_1 \cup A_2 \subset \overline{A_1} \cup \overline{A_2}$, which is a closed set being union of two closed sets. So $\overline{A_1 \cup A_2} \subset \overline{A_1} \cup \overline{A_2}$ (recall the closure of a set is the smallest closed set containing that set so any closed set containing the set contains its closure) . The general case follows from induction.

(b) Note that $\overline{A_i} \subset \overline{\bigcup_{i=1}^{\infty} A_i}$ for each i so $\bigcup_{i=1}^{\infty} \overline{A_i} \subset \overline{\bigcup_{i=1}^{\infty} A_i}$. The converse is false, as can be shown by taking $A_i = \{q_i\}$ to be the ith rational number in some enumeration of $\mathbb{Q} = \{q_1, q_2, \cdots\}$. $\bigcup_{i=1}^{\infty} \overline{A_i} = \bigcup_{i=1}^{\infty} \{q_i\} = \mathbb{Q}$ because points are closed in a metric space (Verify). On the other hand, $\overline{\bigcup_{i=1}^{\infty} A_i} = \overline{\mathbb{Q}} = \mathbb{R}$.

■

2.9

Let $B_r(x)$ denote the ball of radius r centered at x, same as neighborhood in Rudin.

(a) E° is open.

(b) E is open iff $E^\circ = E$.

(c) If $G \subset E$ and G is open, $G \subset E^\circ$, i.e. E° is the largest open set contained in E.

(d) $(E^\circ)^c = \overline{E^c}$.

(e) Do E and $\overline{E}$ always have the same interior?

(f) Do E and E° always have the same closure?

Proof. (a) Let $x \in E^\circ$. Then there exists $r > 0$ s.t. $B_r(x) \subset E$. For each $y \in B_r(x)$, $y \in B_{r-d(x,y)}(y) \subset B_r(x)$ so $B_r(x) \subset E^\circ$.

(b) If $E^\circ = E$, E is open by (a). If E is open, then $E^\circ = E$ by definition (E is open if every point in E is an interior point).

(c) $G = G^\circ \subset E^\circ$. (Verify if $G \subset E$ then $G^\circ \subset E^\circ$)

(d)

$$(E^\circ)^c = \Bigg(\bigcup_{\substack{G \subset E \\ G \text{ open}}} G \Bigg)^c = \bigcap_{\substack{E^c \subset G^c \\ G^c \text{ closed}}} G^c = \overline{E^c}.$$

(e) Let $E = \mathbb{Q}$. $E^\circ = \emptyset$ whereas $(\overline{E})^\circ = \mathbb{R}^\circ = \mathbb{R}$. (Holes have been filled)

(f) Let $E = \mathbb{Q}$. $\overline{E} = \mathbb{R}$ whereas $\overline{E^\circ} = \overline{\emptyset} = \emptyset$.

■

2.10

Proof. To show triangle inequality $d(p,q) \le d(p,s) + d(s,q)$, note that the maximum of the left hand side is 1, and is attained when $p \ne q$. Then $d(p,s)$ and $d(q,s)$ can be be 0 at the same time. d is called a discrete metric on X and (X,d) is called discrete metric space or a space of isolated points.
Observe that $B_{1/2}(x) = \{x\}$ so every point is open. Every subset is a union of points so all subsets are open and hence all are closed. $\bigcup_{x\in K}\{x\}$ is an open over of K so if K were compact, K has to be finite to have a finite subcover. Conversely, every finite subset of X is compact. (Verify). ■

2.11

Proof. (a) $d(0,1) + d(1,2) < d(0,2)$.

(b) It is a metric. Square $\sqrt{|x-y|} \le \sqrt{|x-z|} + \sqrt{|y-z|}$ to verify the triangle inequality.

(c) $d(1,-1) = 0$.

(d) $d(1,\frac{1}{2}) = 0$ and the metric is also nonsymmetric.

(e) It is a metric. Let $f(t) = \frac{t}{1+t}$. Note that f is an increasing function of t on $[0,\infty)$. Then

$$\begin{aligned} f(d(x,y)) \le f(d(x,z) + d(y,z)) &= \frac{d(x,z) + d(y,z)}{1 + d(x,z) + d(y,z)} \\ &= \frac{d(x,z)}{1 + d(x,z) + d(y,z)} + \frac{d(y,z)}{1 + d(x,z) + d(y,z)} \\ &\le f(d(x,z)) + f(d(y,z)). \end{aligned}$$

Note that we prove a more general fact: if d is a metric on X, then so is $\rho = \frac{d}{1+d}$.

■

2.22

$\mathbb{R}^k$ is separable.

Proof. We claim that $\mathbb{Q}^k$ is a countable base for $\mathbb{R}^k$. It suffices to show $\overline{\mathbb{Q}^k} = \mathbb{R}^k$. Let $\mathbf{x} = (x_1, \cdots, x_k) \in \mathbb{R}^k$ and let $B_r(x)$ be an open ball that contains x. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, pick a rational number $q_i \in (x_i - \frac{r}{\sqrt{k}}, x_i + \frac{r}{\sqrt{k}})$ so that $|x_i - q_i| < \frac{r}{\sqrt{k}}$ and let $\mathbf{q} = (q_1, \cdots, q_k) \in \mathbb{Q}^k$. Then

$$|\mathbf{x} - \mathbf{q}| = \sqrt{\sum_{i=1}^{k} |x_i - q_i|^2} < \sqrt{\sum_{i=1}^{k} \frac{r^2}{k}} = r.$$

■

2.23

Every separable metric space has a countable base.

Proof. Let (X, ρ) be a metric space and let D denote the countable dense subset. Let $\mathcal{B} = \{B_r(d)\}_{r\in\mathbb{Q}^+, d\in D}$ be the collection of open balls with rational radius centered in some element in D. The collection D is countable. Let $x \in X$ and G be an open subset of X containing x. So there exists $s > 0$ such that $x \in B_s(x) \subset G$. Since D is dense in X, there exists $d \in D$ such that $\rho(d, x) < \frac{s}{2}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ such that $\rho(d, x) < r < \frac{s}{2}$. Check that $x \in B_r(d) \subset B_r(x) \subset G$. Hence $\mathcal{B}$ is a base. ■

Remark. *Having a countable base is called second countable in point-set topology. This exercise shows that every separable metric space is second countable. In fact you can also show second countable implies separability so in metric spaces the two conditions are equivalent.*