Solution to Homework 4

Haiyu Huang

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Disclaimer: The solution may contain errors or typos so use at your own risk.

2.12

Problem. *Let $K \subset \mathbb{R}$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \cdots$. Prove that K is compact directly from the definition.*

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of K. Let $\alpha_0 \in A$ be the index such that $0 \in U_{\alpha_0}$. Since U_{α_0} is open, there exists $r > 0$ such that $(-r, r) \in U_{\alpha_0}$. This implies for all $n > \frac{1}{r}$, $\frac{1}{n} \in U_{\alpha_0}$. Then choose an open set U_{α_n} in the open cover for each of the $\frac{1}{n} \in K$ where $n \leq \frac{1}{r}$. Adjoining U_{α_0} gives a finite subcover of K. ■

2.16

Problem. *Regards $\mathbb{Q}$ as a metric space with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?*

Proof. Notice that $\mathbb{R}$ is a metric space with distance function $d(x, y) = |x - y|$. $\mathbb{Q} \subset \mathbb{R}$ is a metric space in its own right by restricting d to $\mathbb{Q} \times \mathbb{Q}$. The restricted distance function is called the metric induced on $\mathbb{Q}$ by d and $(\mathbb{Q}, d)$ is a metric subspace of $(\mathbb{R}, d)$. By theorem 2.30, $E \subset \mathbb{Q}$ is open in $\mathbb{Q}$ iff $E = \mathbb{Q} \cap U$ for some open subset of U of $\mathbb{R}$. Similarly, $E \subset \mathbb{Q}$ is closed in $\mathbb{Q}$ iff $E = \mathbb{Q} \cap V$ for some closed subset of V of $\mathbb{R}$ (Verify!). Now E is closed because E is the intersection of a closed set of $\mathbb{R}$ and $\mathbb{Q}$:

$$E = \mathbb{Q} \cap \{p \in \mathbb{R} \mid 2 < p^2 < 3\} = \mathbb{Q} \cap \{p \in \mathbb{R} \mid 2 \leq p^2 \leq 3\} = \mathbb{Q} \cap \Big([-\sqrt{3}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{3}]\Big),$$

where $[-\sqrt{3}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{3}]$ is a closed in $\mathbb{R}$ and the second equality follows from the fact that $\sqrt{2}$ and $\sqrt{3}$ are irrational. E is open because

$$E = \mathbb{Q} \cap \{p \in \mathbb{R} \mid 2 < p^2 < 3\} = \mathbb{Q} \cap \Big((-\sqrt{3}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{3})\Big).$$

By theorem 2.33, E is compact in $\mathbb{Q}$ iff E is compact in $\mathbb{R}$. However E is not closed in $\mathbb{R}$ ($\overline{E} =$?) and hence not compact by Heine-Borel. $E \subset [-2, 2]$ is obviously bounded. ■

2.18

Problem (Optional)**.** *Is there a nonempty perfect set in* $\mathbb{R}$ *which contains no rational number?*

Proof. Let E_0 be a interval with your favorite irrational endpoints, say $[-e, e]$. Let $\{q_1, q_2, \cdots\}$ be the enumeration of rational numbers in E_0. We perform similar construction as in the construction of Cantor set, except in stage k we exclude the kth rational number q_k using a subinterval with irrational endpoints. Now assume inductively that E_n has been constructed such that E_n is a pairwise disjoint union of closed interval with irrational endpoints, each of length at most $3^{-n} \cdot 2e$ and E_n does not contain q_k for $k \leq n$. Construct F_{n+1} by removing the middle third of each of the intervals in E_n. If $q_{n+1} \not\in F_{n+1}$, then let $E_{n+1} = F_{n+1}$; otherwise q_{n+1} is contained in some interval $[a, b]$, where $a, b \in \mathbb{R} \setminus \mathbb{Q}$. Let $\epsilon > 0$ be a irrational number less than $\min(q_{n+1} - a, b - q_{n+1})$ (Why?) and let $E_{n+1} = F_{n+1} \setminus (q_{n+1} - \epsilon, q_{n+1} + \epsilon)$ so that $q_{n+1} \not\in E_{n+1}$. E_n is closed and bounded so E_n form a nested sequence of nonempty compact sets. Hence $P = \bigcap_{n=1}^{\infty} E_n$ is a nonempty compact set. It contains no rational number by construction. Let $x \in P$, then for each n there exists a unique interval $I_n = [a_n, b_n]$ among the disjoint intervals whose union is E_n such that $x \in I_n$. Let $y_n = a_n$ if $x \neq a_n$ and $y_n = b_n$ if $x = a_n$ so that $y_n \in P$ and $|y_n - x| < 3^{-n} \cdot 2e$. Hence $x \in P'$ and P is perfect. ■

2.19

Problem. *Prove that every connected metric space with at least two points is uncountable.*

Proof. Let x, y be two distinct points in X. For every $r \in (0, d(x, y))$, there exists $z \in X$ such that $d(x, z) = r$, otherwise $B_r(x)$ and $\{p \in X \mid d(x, p) > r\}$ are nonempty separated sets whose union is X, contradicting the hypothesis that X is connected. Hence we've constructed a subset of X with a bijection with $[0, d(x, y)]$, which is uncountable. So X is uncountable. ■

2.20

Problem. *Are closures and interiors of connected sets always connected?*

Proof. Let A be connected subset of X. We claim if $A \subset B \subset \overline{A}$ then B is connected. Suppose on the contrary $B = E \cup F$, where E and F are nonempty separated sets. To arrive at a contradiction, we would like $\tilde{E} = A \cap E$ and $\tilde{F} = A \cap F$ to be nonempty separated sets whose union is A. It is obviously $\tilde{E}$ and $\tilde{F}$ are separated and $\tilde{E} \cup \tilde{F} = A$. It remains to check $\tilde{E}$ and $\tilde{F}$ are nonempty. Suppose $\tilde{F} = \emptyset$ and so $A \subset E$. Then since $F \neq \emptyset$, F contains a limit point of A and so contains a limit point of E, contradicting $\overline{E} \cap F = \emptyset$ because E and F are separated. Therefore, $\tilde{E}$ and $\tilde{F}$ are nonempty separated sets whose union is A, implying A is not connected, a contradiction.
The interior of a connected set may fail to be connected. Take, for example, the union of two closed disks of radius 1 center at $(1, 0)$ and $(-1, 0)$ in $\mathbb{R}^2$, the interior of which are disjoint open disks. ■

2.21

Problem. *Every convex subset of* $\mathbb{R}^k$ *is connected.*

Proof. Let $E \subset \mathbb{R}^k$ be a convex set. Suppose on the contrary that there exists nonempty subsets $A, B \subset E$ such that $A \cup B = E$ and A, B are separated. Let $a \in A$, $b \in B$ and define $p(t) = (1-t)a + tb$ for $t \in \mathbb{R}$. Let $A_0 = p^{-1}(A)$ and $B_0 = p^{-1}(B)$. Assume towards contradiction that $\overline{A}_0 \cap B_0 \neq \emptyset$. Then there exists $s \in \mathbb{R}$ such that $p(s) \in B$ and for every $r > 0$, there exists $t \in \mathbb{R}$ such that $|s-t| < r$ and $p(t) \in A$. Now

$$|p(s), p(t)| = |s-t||a-b| \leq (|a|+|b|)|s-t| < (|a|+|b|)r.$$

Since r is arbitrary and $p(t) \in A$, $p(s) \in \overline{A} \cap B$, contradicting that A and B are separated. Hence A_0 and B_0 are separated. Since E is convex, $p(t) \in E = A \cup B$ for all $t \in [0,1]$. This implies $[0,1] \subset A_0 \cup B_0$. So $[0,1] = U \cup V$, where $U = A_0 \cap [0,1]$ and $V = B_0 \cap [0,1]$. Note that U,V are nonempty separated set, implying that $[0,1]$ is not connected, a contradiction. Therefore, every convex subset of $\mathbb{R}^k$ is connected. ■

2.24

Problem. *Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.*

Proof. For each $\delta > 0$, we construct the following set: pick $x_1 \in X$. Having chosen $x_1, \cdots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \cdots, j$. This process must stop after a finite number of steps, otherwise for any $x \in X$, $B_{\frac{\delta}{2}}(x)$ contains at most one point of the infinite set, and hence no point could be a limit point of this set, contradicting the hypothesis. So it follows that for each $\delta = \frac{1}{n} > 0$, X is covered by open balls of radius $\frac{1}{n}$ centered at the finitely many points we constructed $x_{n1}, x_{n2}, \cdots, x_{nm_n}$ for some m_n depending on n , i.e. $X = \bigcup_{j=1}^{m_n} B_{\frac{1}{n}}(x_{nj})$. Consider $D = \{x_{nj}, 1 \leq j \leq m_n, n = 1, 2, \cdots\}$. D is countable since D is a countable union of finite sets. Let $x \in X$ and $r > 0$. Then there exists $n \in \mathbb{N}$ such that $r > \frac{1}{n}$ by Archimedean property of $\mathbb{R}$. $X = \bigcup_{j=1}^{m_n} B_{\frac{1}{n}}(x_{nj})$ implies $x \in B_{\frac{1}{n}}(x_{nj})$ for some $1 \leq j \leq m_n$. So $x_{nj} \in B_r(x)$. Therefore D is a countable dense subset of X and X is separable. ■

2.25

Problem. *Prove that every compact metric space K has a countable base, and that K is therefore separable.*

Proof. Let $n \in \mathbb{N}$. $\cup_{x \in X} B_{\frac{1}{n}}(x)$ is an open cover of K and since K is compact, $K \subset \bigcup_{j=1}^{m_n} B_{\frac{1}{n}}(x_{nj})$. Similar to the previous problem, $D = \{x_{nj}, 1 \leq j \leq m_n, n = 1, 2, \cdots\}$ is a countable dense subset of X and $\mathcal{B} = \{B_{\frac{1}{n}}(x_{nj}), 1 \leq j \leq m_n, n = 1, 2, \cdots\}$ is a countable base for K. ■

2.29

Problem. *Prove that every open set in* $\mathbb{R}$ *is the union of an at most countable collection of disjoint segments.*

Proof. If U is open, for each $x \in U$ we construct the largest interval containing x: consider the collection $\mathcal{I}_x$ of all open intervals I such that $x \in I \subset U$. It is easy to check that the union of any family of open intervals containing a point is common is again an open interval (Verify!), and hence $J_x = \bigcup_{I \in \mathcal{I}_x} I$ is an open interval: it is the largest element of $\mathcal{I}_x$. If $x, y \in U$ then either $J_x = J_y$ or $J_x \cap J_y = \emptyset$, for otherwise $J_x \cup J_y$ would be a larger open interval than J_x in $\mathcal{I}_x$. Let $\mathcal{J} = \{J_x : x \in U\}$, where the members of $\mathcal{J}$ are disjoint, and $U = \bigcup_{J \in \mathcal{J}} J$. For each $J \in \mathcal{J}$, pick a rational number $f(J) \in J$. This map $f : \mathcal{J} \to \mathbb{Q}$ thus defined is an injection, for if $J \neq J'$ then $J \cap J' = \emptyset$; therefore $\mathcal{J}$ is countable. ∎