Solution to Homework 6

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Disclaimer: The solution may contain errors or typos so use at your own risk.

3.4

Problem. *Find the upper and lower limits of the sequence* $\{s_n\}$ *defined by*

$$s_1 = 0; \qquad s_{2m} = \frac{s_{2m-1}}{2}; \qquad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Proof. $s_{2m+2} = \frac{s_{2m+1}}{2} = \frac{1}{4} + \frac{1}{2}s_{2m}$. With this recurrence, it can be shown by induction that $s_{2m} = 2^{-1} - 2^{-k}$. So it follows that

$$s_n = \begin{cases} 2^{-1} - 2^{-k} & n = 2k \\ 1 - 2^{-k} & n = 2k+1 \end{cases}.$$

Now it is easy to see that $\limsup_{n\to\infty} s_n = \lim_{k\to\infty} s_{2k+1} = 1$ and $\liminf_{n\to\infty} s_n = \lim_{k\to\infty} s_{2k} = 2^{-1}$. ∎

3.5

Problem. *For any two real sequences* $\{a_n\}$, $\{b_n\}$, *prove that*

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

Proof. If either $\limsup a_n = \infty$ or $\limsup b_n = \infty$, there is nothing to prove. So assume $\limsup a_n < \infty$ and $\limsup b_n < \infty$. Let $\epsilon > 0$. There exists $N_1 > 0$ such that $a_n < \limsup a_n + \frac{\epsilon}{2}$ for all $n \ge N_1$ and there exists $N_2 > 0$ such that $b_n < \limsup b_n + \frac{\epsilon}{2}$ for all $n \ge N_2$. So for $n \ge N = \max(N_1, N_2)$, $a_n + b_n \le \limsup a_n + \limsup b_n + \epsilon$. Therefore, $\limsup(a_n + b_n) \le \limsup a_n + \limsup b_n$.
Alternatively, note that $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{k\ge n} x_k = \inf_{n\ge 0} \sup_{m\ge n} x_m$. Fix $n > 0$. Observe that for all $m \ge n$

$$(a_m + b_m) \le \sup_{m\ge n} a_m + \sup_{m\ge n} b_m.$$

Take the sup over all $m \ge n$,

$$\sup_{m\ge n}(a_m + b_m) \le \sup_{m\ge n} a_m + \sup_{m\ge n} b_m.$$

Taking the limit as $n \to \infty$ (Why does the limit exists?) ,

$$\limsup_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty}\sup_{m\geq n}(a_m+b_m) \leq \lim_{n\to\infty}\sup_{m\geq n} a_m + \lim_{n\to\infty}\sup_{m\geq n} b_m = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

■

3.6

Problem. *Investigate the behavior of* $\sum a_n$ *if*

(a) $a_n = \sqrt{n+1} - \sqrt{n}$;

(b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$;

(c) $a_n = (\sqrt[n]{n}-1)^n$;

(d) $a_n = \frac{1}{1+z^n}$, *for* $z \in \mathbb{C}$.

Proof. (a) The partial sum $s_n = \sqrt{n+1}-1 \to \infty$ so $\sum a_n$ diverges.

(b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \leq \frac{1}{2n^{3/2}}$, which converges as a p-series with $p = 3/2 > 1$. So $\sum a_n$ converges by comparison test.

(c) $\limsup \sqrt[n]{a_n} = \lim(\sqrt[n]{n}-1) = 0 < 1$, so the series converges by root test.

(d) If $|z| > 1$, then

$$\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty}\left|\frac{1+z^n}{1+z^{n+1}}\right| = \limsup_{n\to\infty}\left|\frac{z^{-n}+1}{z^{-n}+z}\right| = \frac{1}{|z|} < 1,$$

where $\limsup|z^{-n}| = \lim|z|^{-n} \to 0$ as $|z| > 1$. So $\sum a_n$ converges by ratio test. If $|z| \leq 1$, then by triangle inequality

$$|a_n| = \frac{1}{|1+z^n|} \geq \frac{1}{1+|z|^n} \geq \frac{1}{2}.$$

Since $a_n \not\to 0$, $\sum a_n$ diverges.

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3.7

Problem. *Prove that the convergence of* $\sum a_n$ *implies the convergences of* $\sum \frac{\sqrt{a_n}}{n}$ *if* $a_n \geq 0$.

Proof. By Cauchy Schwarz inequality,

$$\sum \frac{\sqrt{a_n}}{n} \leq \left(\sum a_n\right)^{1/2} \cdot \left(\sum \frac{1}{n^2}\right)^{1/2},$$

which is the product of two convergent series. ■

Problem 2

Assume that the series $\sum_{n=1}^{\infty} a_n^2$ converges. Prove that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

Proof. Let $b_n = a_n^2 \geq 0$. By the previous problem, $\sum a_n^2 = \sum b_n$ converges implies that $\sum \frac{\sqrt{b_n}}{n} = \sum \frac{|a_n|}{n}$ converges. So $\sum \frac{a_n}{n}$ converges. ■

3.8

Problem. *If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.*

Proof. Let $A_n = \sum_{k=1}^{n} a_k (A_0 = 0)$, so that $a_n = A_n - A_{n-1}$ for $n = 1, 2, \cdots$. A_n converges implies A_n is bounded. Let M be an upper bound for A_n. $\{b_n\}$ is monotonic and bounded implies the convergence of $\{b_n\}$. Since $\{b_n\}$ and $\{A_n\}$ converges, the product $\{b_n A_n\}$ also converges. In particular both $\{b_n\}$ and $\{b_n A_n\}$ are Cauchy sequences. Let $\epsilon > 0$. Choose N sufficiently large so that the following inequalities hold $\forall m, n \geq N$:

$$|b_n A_n - b_m A_m| < \frac{\epsilon}{2}; \qquad |b_m - b_n| < \frac{\epsilon}{2M}.$$

Then if $n > m \geq N$, by Abel's summation by parts formula

$$\sum_{k=m+1}^{n} a_k b_k = b_n A_n - b_m A_m + \sum_{k=m}^{n-1} (b_k - b_{k+1}) A_k.$$

Since the sequence $\{b_k\}$ is monotonic, we have

$$\left| \sum_{k=m}^{n-1} (b_k - b_{k+1}) A_k \right| \leq M \sum_{k=m}^{n-1} |b_k - b_{k+1}| = M \left| \sum_{k=m}^{n-1} (b_k - b_{k+1}) \right| = M |b_m - b_n| < M \cdot \frac{\epsilon}{2M} = \frac{\epsilon}{2}.$$

Therefore,

$$\left| \sum_{k=m+1}^{n} a_k b_k \right| \leq |b_n A_n - b_m A_m| + \left| \sum_{k=m}^{n-1} (b_k - b_{k+1}) A_k \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\sum a_n b_n$ is Cauchy and thus converges by Cauchy criterion. ■

3.14

Problem. *If* $\{s_n\}$ *is a complex sequence, define its arithmetic mean* σ_n *by*

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \qquad (n = 0, 1, 2, \cdots).$$

(a) *If* $\lim s_n = s$, *prove that* $\lim \sigma_n = s$.

(b) *Construct a sequence* $\{s_n\}$ *which does not converge, although* $\lim \sigma_n = 0$.

(c) *Can it happen that* $s_n > 0$ *for all* n *and that* $\limsup s_n = \infty$, *although* $\lim \sigma_n = 0$.

(d) *Put* $a_n = s_n - s_{n-1}$, *for* $n \geq 1$. *Show that* $s_n - \sigma_n = \frac{1}{n+1}\sum_{k=1}^{n} k a_k$. *Assume* $\lim(n a_n) = 0$ *and that* $\{\sigma_n\}$ *converges. Prove that* $\{s_n\}$ *converges.*

Proof. (a) Let $\epsilon > 0$. Let $t_k = s_k - s$.

$$\begin{aligned} |\sigma_n - s| &= \left|\frac{s_0 + s_1 + \cdots + s_n}{n+1} - s\right| \\ &= \left|\frac{(s_0 - s) + (s_1 - s) + \cdots + (s_n - s)}{n+1}\right| \\ &= \left|\frac{t_0 + t_1 + \cdots + t_n}{n+1}\right|. \end{aligned}$$

Since $\lim s_n = s$, $\exists N > 0$ such that $|t_n| < \epsilon$ if $n \geq N$. Now

$$\begin{aligned} |\sigma_n - s| &= \left|\frac{t_0 + t_1 + \cdots + t_n}{n+1}\right| \\ &= \left|\frac{t_0 + t_1 + \cdots + t_N}{n+1} + \frac{t_{N+1} + \cdots + t_n}{n+1}\right| \\ &\leq \frac{|t_0| + |t_1| + \cdots + |t_N|}{n+1} + \frac{|t_{N+1}| + \cdots + |t_n|}{n+1} \end{aligned}$$

Note that

$$\frac{|t_{N+1}| + \cdots + |t_n|}{n+1} < \frac{n-N}{n+1} \cdot \epsilon < \epsilon$$

Moreover, by Archimedean property of $\mathbb{R}$, $\exists N'$ such that $\frac{|t_0| + |t_1| + \cdots + |t_N|}{N'+1} < \epsilon$. Now if $n \geq \max\{N, N'\}$, then

$$|\sigma_n - s| \leq \frac{|t_0| + |t_1| + \cdots + |t_N|}{n+1} + \frac{|t_{N+1}| + \cdots + |t_n|}{n+1} < 2\epsilon.$$

Hence $\lim \sigma_n = s$.

(b) Let $s_n = (-1)^n$. Then $\sigma_n = 0$ if n is odd and $\sigma_n = \frac{1}{n+1}$ if n is even. Thus $\sigma_n \to 0$ even though s_n does not converge.

(c) Heuristically, we need to construct a positive sequence $\{s_n\}$ with a subsequence goes to infinity, but $\frac{\sum_{k=0}^{n} s_k}{n+1} \to 0$. For example, this can be achieved if $\sum s_n$ grows slower than $\sqrt{n}$ with a subsequence grows as $n^{1/4}$. Let $s_n = \begin{cases} k & n = k^4 \\ \frac{1}{n^2} & n \neq k^4 \end{cases}$. The for $k^4 \leq n < (k+1)^4$,

$$\begin{aligned}\sigma_n &\leq \frac{1}{n+1}\sum_{m=1}^{n}\frac{1}{m^2} + \frac{1}{n+1}\sum_{m=1}^{k}\frac{1}{m} \\ &= \frac{1}{n+1}\sum_{m=1}^{n}\frac{1}{m^2} + \frac{1}{n+1}\cdot\frac{k(k+1)}{2}.\end{aligned}$$

The first $\frac{1}{n+1}\sum_{m=1}^{n}\frac{1}{m^2}$ goes to 0 by part (a) and for the second term, $\frac{1}{n+1}\cdot\frac{k(k+1)}{2} \leq \frac{1}{k^4}\cdot\frac{k(k+1)}{2} \leq \frac{1}{2k^2} \to 0$ as $k \to \infty$. As $n \to \infty$, $k \to \infty$ since $(k+1)^4 > n$. Therefore, $\sigma_n \to 0$ as $n \to \infty$ whereas $s_{n=k^4} = n^{1/4} \to \infty$.

(d) Let $a_0 = s_0$ so that

$$\begin{aligned}s_n - \sigma_n &= \frac{(n+1)s_n - \sum_{k=0}^{n} s_k}{n+1} \\ &= \frac{(ns_n - ns_{n-1}) + ((n-1)s_{n-1} - (n-1)s_{n-2}) + \cdots + (s_1 - s_0)}{n+1} \\ &= \frac{1}{n+1}\sum_{k=1}^{n} ka_k.\end{aligned}$$

If $na_n \to 0$, then by part (a) the average of $na_n \to 0$, which is the right hand side of the above equation. Therefore $s_n - \sigma_n \to 0$ and so s_n converges.

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Problem 1

Let $f : \mathbb{N} \to \mathbb{R}$ and $||f|| = \left(\sum_{n=1}^{\infty} |f(n)|^2\right)^{1/2}$. Define Let $l^2 = \{f : \mathbb{N} \to \mathbb{R} : ||f|| < \infty\}$. For two sequences $f, g \in l^2$, define $d(f,g) = ||f - g||$.

(i) Show that the distance is well-defined and that l^2 with this distance is a metric space;

(ii) For each $j \geq 1$, consider the sequence e_j whose terms are all equal to 0 except for the jth term which is 1. Show that for each $j \geq 1$, e_j is an element in l^2 and show that the sequence $\{e_j\}_{j\geq 1}$ is not Cauchy in l^2;

(iii) In the metric space consider the closed unit ball of center the zero sequence $K = \{f \in l^2 : ||f|| \leq 1\}$. Show that K is closed and bounded but not compact by exhibiting a sequence in K that does not have a convergent subsequence.

Proof. (i) To show the distance function is well-defined, it suffices to show $f + g \in l^2$ for $f, g \in l^2$. Observe that

$$|f+g|^2 \leq (|f| + |g|)^2 \leq \left(2\max(|f|,|g|)\right)^2 \leq 4(|f|^2 + |g|^2).$$

It is clear that $d(f,f)=0$, $d(f,g)>0$ if $f\neq g$, and $d(f,g)=d(g,f)$ for $f,g\in l^2$. For triangle inequality, note that $|f+g|^2\leq(|f|+|g|)|f+g|=|f||f+g|+|g||f+g|$. By Cauchy Schwarz inequality,

$$\begin{aligned}\sum_{n=1}^{\infty}|f(n)+g(n)|^2 &\leq \sum_{n=1}^{\infty}|f(n)||f(n)+g(n)|+\sum_{n=1}^{\infty}|g(n)||f(n)+g(n)|\\ &\leq\left(\sum_{n=1}^{\infty}|f(n)|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}|f(n)+g(n)|^2\right)^{1/2}+\left(\sum_{n=1}^{\infty}|g(n)|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}|f(n)+g(n)|^2\right)^{1/2}\\ &=(||f||+||g||)\left(\sum_{n=1}^{\infty}|f(n)+g(n)|^2\right)^{1/2}.\end{aligned}$$

Therefore, $||f+g||=\left(\sum_{n=1}^{\infty}|f(n)+g(n)|^2\right)^{1/2}\leq||f||+||g||$.

(ii) For each $j\geq1$, $||e_j||=1$ so $e_j\in l^2$. $\{e_j\}_{j\geq1}\in l^2$ is not Cauchy because $||e_i-e_j||=\sqrt{2}$ for all $i\neq j$.

(iii) K is bounded by definition. Let $f_n\in l^2$ and suppose $f_n\to f$. Let $\epsilon>0$. Then there exists N such that $||f_n-f||<\epsilon$ for $n\geq N$. So $||f||\leq||f-f_N||+||f_N||<\epsilon+1$. Hence $||f||\leq1$ and $f\in l^2$. This shows that K is closed. Observe $\{e_j\}_{j\geq1}$ defined in part (ii) is a sequence in K, which does not have a convergent subsequence for the same reason that $||e_i-e_j||=\sqrt{2}$ for all $i\neq j$.

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Problem 3

Let $\{a_n\}$ be a sequence of monotonically decreasing positive numbers with the property that $a_n\geq10a_{2n}$ for all $n\in\mathbb{N}$. Prove that $\sum_{n=1}^{\infty}a_n$ converges.

Proof. Note that $a_{2^n}\leq10^{-1}a_{2^{n-1}}\leq10^{-n}a_1$ and for all $2^k\leq n<2^{k+1}$, $a_{2^k}\geq a_n$ since a_n is monotonically decreasing.

$$\begin{aligned}\sum_{n=1}^{\infty}a_n &= a_1+(a_2+a_3)+(a_4+a_5+a_6+a_7)+\cdots\\ &\leq a_1+(a_2+a_2)+(a_4+a_4+a_4+a_4)+\cdots\\ &=a_1+2a_2+4a_4+8a_8+\cdots\\ &\leq a_1+\frac{2}{10}a_1+\frac{2^2}{10^2}a_1+\frac{2^3}{10^3}a_1+\cdots\\ &=a_1\sum_{k=0}^{\infty}\left(\frac{1}{5}\right)^k\\ &=\frac{5}{4}a_1\\ &<\infty.\end{aligned}$$

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