3.9

Problem. Find the radius of convergence of each of the following power series:

\[(a) \sum n^3 z^n, \quad (b) \sum \frac{2^n}{n!} z^n, \quad (c) \sum \frac{2^n}{n^2} z^n, \quad (d) \sum \frac{n^3}{3^n} z^n.\]

Solution. (a) Let \(a_n = n^3, \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \frac{n+1}{2} = \infty\) implies that the radius of convergence is 1.

(b) Let \(a_n = \frac{2^n}{n!}, \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \frac{n+1}{2n} = \infty\) implies that the radius of convergence is infinite.

(c) Let \(a_n = \frac{2^n}{n^2}, \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \lim_{n \to \infty} \frac{(n+1)^2}{2n^2} = \frac{1}{2}\) implies that the radius of convergence is \(\frac{1}{2}\).

(d) Let \(a_n = \frac{n^3}{3^n}, \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \lim_{n \to \infty} 3 \left(\frac{n}{n+1}\right)^3 = 3\) implies that the radius of convergence is 3.

3.10

Problem. Suppose that the coefficients of the power series \(\sum a_n z^n\) are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

Proof. Notice that when \(|z| > 1, a_n\) does not converge to zero since infinitely many terms are nonzero, which implies the series diverges. Thus, the radius of convergence is at most 1.

3.11

(a)

Proof. First, if \(\{a_n\}\) is unbounded, then \(\frac{a_n}{1+a_n}\) does not converge to zero as \(n \to \infty\), which implies the series diverges.

Next, if \(\{a_n\}\) is upper bounded by some \(M > 0\), then

\(\frac{a_n}{1+a_n} \geq \frac{a_n}{1+M}.\)

By comparison test, we have the series is divergence.
(b)

**Proof.** Notice $s_n$ is strictly increasing as $n$ goes up. For $k \geq 1$, we have

$$\frac{a_{N+1} + \ldots + a_{N+k}}{s_{N+1} + \ldots + s_{N+k}} \geq \frac{a_{N+1} + \ldots + a_{N+k}}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}.$$

Since $S_n$ goes to $\infty$ as $n \to \infty$, for any $N$, we can find $k$ large enough such that $1 - \frac{s_N}{s_{N+k}} > \frac{1}{2}$, which implies the partial sums of the series $\sum \frac{a_n}{s_n}$ is not a Cauchy sequence. It follows that the series diverges. \(\blacksquare\)

(c)

**Proof.** Notice that for $n \geq 2$,

$$\frac{a_n}{s_n^2} \leq \frac{a_n}{s_n s_{n-1}} = \frac{1}{s_{n-1}} - \frac{1}{s_n}.$$

Thus,

$$\sum \frac{a_n}{s_n^2} \leq \sum \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{1}{a_1}.$$

\(\blacksquare\)

(d) $\sum \frac{a_n}{1+na_n}$ can be either convergent or divergent.

- If $\{na_n\}$ is bounded above, it converges.
- If $a_n = \frac{1}{n(\log(n))^p}$, then $\sum \frac{a_n}{1+na_n} = \sum \frac{1}{n(\log(n))^{p+1} + n} < \frac{1}{n(\log(n))^p}$, which implies $\sum \frac{a_n}{1+na_n}$ converges.

For the second series, notice

$$\sum \frac{a_n}{1+n^2a_n} = \sum \frac{1}{a_n + n^2} \leq \sum \frac{1}{n^2}.$$

It follows that $\sum \frac{a_n}{1+n^2a_n}$ converges.
3.13

**Problem.** Prove that the Cauchy product of two absolutely convergent series converges absolutely.

**Proof.** Let \( S_n = \sum_{k=0}^{n} a_n \), \( T_n = \sum_{k=0}^{n} b_n \) and \( U_n = \sum_{k=0}^{n} \sum_{l=0}^{k} a_l b_{k-l} \). Let \( \lim_{n \to \infty} S_n = S \) and \( \lim_{n \to \infty} T_n = T \). Since \( \{a_n\} \) and \( \{b_n\} \) are two absolutely convergent series, we can assume without lost of generalization that \( a_n, b_n \geq 0 \) for all \( n \in \mathbb{N} \). Assume \( a_{-1} = T_{-1} = 0 \). We have

\[
U_n = \sum_{k=0}^{n} \sum_{l=0}^{k} a_l b_{k-l}
\]

\[
= \sum_{k=0}^{n} \sum_{l=0}^{k} a_l (T_{k-l} - T_{k-l-1})
\]

\[
= \sum_{k=0}^{n} \sum_{j=0}^{k} a_{k-j} (T_j - T_{j-1})
\]

\[
= \sum_{k=0}^{n} \sum_{j=0}^{k} (a_{k-j} - a_{k-j-1}) T_j
\]

\[
= \sum_{j=0}^{n} \sum_{k=j}^{n} (a_{k-j} - a_{k-j-1}) T_j
\]

\[
= \sum_{j=0}^{n} a_{n-j} T_j
\]

\[
\leq T \sum_{m=0}^{n} a_m
\]

\[
= TS_n \leq TS.
\]

Thus, we have shown that \( U_n \) is bounded above which implies the increasing sequence \( U_n \) converges. \( \blacksquare \)