Solution to Homework 7

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3.9

Problem. *Find the radius of convergence of each of the following power series:*

$$(a)\sum n^3 z^n, \qquad (b)\sum \frac{2^n}{n!} z^n,$$
$$(c)\sum \frac{2^n}{n^2} z^n, \qquad (d)\sum \frac{n^3}{3^n} z^n.$$

Solution. *(a) Let* $a_n = n^3$. $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$ *implies that the radius of convergence is* 1.
(b) Let $a_n = \frac{2^n}{n!}$. $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{n+1}{2} = \infty$ *implies that the radius of convergence is infinite.*
(c) Let $a_n = \frac{2^n}{n^2}$. $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{(n+1)^2}{2n^2}$ *implies that the radius of convergence is* $\frac{1}{2}$.
(d) Let $a_n = \frac{n^3}{3^n}$. $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} 3\left(\frac{n}{n+1}\right)^3 = 3$ *implies that the radius of convergence is* 3.

3.10

Problem. *Suppose that the coefficients of the power series* $\sum a_n z^n$ *are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most* 1.

Proof. Notice that when $|z| > 1$, a_n does not converge to zero since infinitely many terms are nonzero, which implies the series diverges. Thus, the radius of convergence is at most 1. ■

3.11

(a)

Proof. First, if $\{a_n\}$ is unbounded, then $\frac{a_n}{1+a_n}$ does not converge to zero as $n \to \infty$, which implies the series diverges.
Next, if $\{a_n\}$ is upper bounded by some $M > 0$, then

$$\frac{a_n}{1+a_n} \geq \frac{a_n}{1+M}.$$

By comparison test, we have the series is divergence. ■

(b)

Proof. Notice s_n is strictly increasing as n goes up. For $k \geq 1$, we have

$$\begin{aligned}\frac{a_{N+1}}{s_{N+1}} + \dots \frac{a_{N+k}}{s_{N+k}} &\geq \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} \\ &= \frac{s_{N+k} - s_N}{s_{N+k}} \\ &= 1 - \frac{s_N}{s_{N+k}}.\end{aligned}$$

Since S_n goes to ∞ as $n \to \infty$, for any N, we can find k large enough such that $1 - \frac{s_N}{s_{N+k}} > \frac{1}{2}$, which implies the partial sums of the series $\sum \frac{a_n}{s_n}$ is not a Cauchy sequence. It follows that the series diverges. ∎

(c)

Proof. Notice that for $n \geq 2$,

$$\begin{aligned}\frac{a_n}{s_n^2} &\leq \frac{a_n}{s_n s_{n-1}} \\ &= \frac{1}{s_{n-1}} - \frac{1}{s_n}\end{aligned}$$

Thus,

$$\sum \frac{a_n}{s_n^2} \leq \sum_{n=2}^{\infty} \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{1}{a_1}.$$

∎

(d) $\sum \frac{a_n}{1+na_n}$ can be either convergent or divergent.

- If $\{na_n\}$ is bounded above, it converges.
- If $a_n = \frac{1}{n(\log(n))^p}$, then $\sum \frac{a_n}{1+na_n} = \sum \frac{1}{n(\log(n))^p + n} < \frac{1}{n(\log(n))^p}$, which implies $\sum \frac{a_n}{1+na_n}$ converges.

For the second series, notice

$$\sum \frac{a_n}{1+n^2 a_n} = \sum \frac{1}{\frac{1}{a_n} + n^2} \leq \sum \frac{1}{n^2}.$$

It follows that $\sum \frac{a_n}{1+n^2 a_n}$ converges.

3.13

Problem. *Prove that the Cauchy product of two absolutely convergent series converges absolutely.*

Proof. Let $S_n = \sum_{k=0}^{n} a_n$, $T_n = \sum_{k=0}^{n} b_n$ and $U_n = \sum_{k=0}^{n} \sum_{l=0}^{k} a_l b_{k-l}$. Let $\lim_{n\to\infty} S_n = S$ and $\lim_{n\to\infty} T_n = T$. Since $\{a_n\}$ and $\{b_n\}$ are two absolutely convergent series, we can assume witout lost of generalization that $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$. Assume $a_{-1} = T_{-1} = 0$. We have

$$
\begin{aligned}
U_n &= \sum_{k=0}^{n} \sum_{l=0}^{k} a_l b_{k-l} \\
&= \sum_{k=0}^{n} \sum_{l=0}^{k} a_l (T_{k-l} - T_{k-l-1}) \\
&= \sum_{k=0}^{n} \sum_{j=0}^{k} a_{k-j} (T_j - T_{j-1}) \\
&= \sum_{k=0}^{n} \sum_{j=0}^{k} (a_{k-j} - a_{k-j-1}) T_j \\
&= \sum_{j=0}^{n} \sum_{k=j}^{n} (a_{k-j} - a_{k-j-1}) T_j \\
&= \sum_{j=0}^{n} a_{n-j} T_j \\
&\leq T \sum_{m=0}^{n} a_m \\
&= T S_n \leq T S.
\end{aligned}
$$

Thus, we have shown that U_n is bounded above which implies the increasing sequence U_n converges. ■