Solution to Homework 8

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Disclaimer: The solution may contain errors or typos so use at your own risk.

Problem. *Assume $f : X \to Y$ is Lipschitz continuous, i.e. there exists $C \geq 0$ such that*

$$d_Y(f(p), f(q)) \leq C d_X(p,q)$$

for all $p, q \in X$. Prove that f is uniformly continuous.

Proof. For all $\epsilon > 0$, let $\delta = \frac{\epsilon}{C}$. Then $d_Y(f(p), f(q)) \leq C d_X(p,q) < \epsilon$ whenever $d_X(p,q) < \delta$. ■

4.2

Problem. *If f is a continuous mapping of a metric space X into a metric space Y, prove that*

$$f(\overline{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

Proof. $f(E) \subset \overline{f(E)}$, so $E \subset f^{-1}\left(f(E)\right) \subset f^{-1}\left(\overline{f(E)}\right)$. Since $\overline{f(E)}$ is closed and f is continuous, $f^{-1}\left(\overline{f(E)}\right)$ is closed so $\overline{E} \subset f^{-1}\left(\overline{f(E)}\right)$. Hence $f(\overline{E}) \subset \overline{f(E)}$.
You can choose any continuous function on the real line that approaches a value at infinity. For example, let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{1+x^2}$ and $E = [0, \infty)$. Then $f(\overline{E}) = f(E) = (0,1]$, but $\overline{f(E)} = [0,1]$.

Remark. *The converse is also true: if $f(\overline{E}) \subset \overline{f(E)}$ for every $E \subset X$, then f is continuous. Try to prove it.*

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4.3

Problem. *Let f be a continuous real function on a metric space X. Let $Z(f)$ be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.*

Proof. $\{0\}$ is closed in $\mathbb{R}$ so $Z(f) = f^{-1}(\{0\})$ is closed by the continuity of f. ■

4.4

Problem. *Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$.*

Proof. By Problem 4.2, $f(X) = f(\overline{E}) \subset \overline{f(E)}$. So $f(X) = \overline{f(E)}$. Define $\psi : X \to \mathbb{R}$ by $\psi(p) = d_Y(f(p), g(p))$. The zero set of ψ, $Z(\psi) = \{p \in X : \psi(p) = 0\} = \{p \in X : f(p) = g(p)\} \supset E$. ψ is continuous: Let $\epsilon > 0$, there exists $\delta_1 > 0$ such that $|p - q| < \delta_1 \implies |f(p) - f(q)| < \frac{\epsilon}{2}$, and there exists $\delta_2 > 0$ such that $|p - q| < \delta_2 \implies |g(p) - g(q)| < \frac{\epsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$. If $|p - q| < \delta$,

$$|\psi(p) - \psi(q)| = \left| d_Y(f(p), g(p)) - d_Y(f(q), g(q)) \right| \le d_Y(f(p), f(q)) + d_Y(g(p), g(q)) < \epsilon.$$

Hence ψ is continuous and $Z(\psi)$ is closed. Since $E \subset Z(\psi)$, $X = \overline{E} \subset Z(\psi)$, i.e. $g(p) = f(p)$ for all $p \in X$. ■

4.8

Problem. *Let f be a real uniformly continuous function on the bounded set E in $\mathbb{R}$. Prove that f is bounded on E. Show that the conclusion is false if boundedness of E is omitted from the hypothesis.*

Proof. Since E is bounded, we would like to take the closure of E so that E is compact. To use the image of $\overline{E}$ under f is compact, we have to define f on the limit points.

Lemma (Continuous Extension)**.** *Let X be a metric space and $E \subset X$. If $f : E \to \mathbb{R}$ is uniformly continuous, then there exists a unique continuous extension $\tilde{f} : \overline{E} \to \mathbb{R}$ such that $\tilde{f}|_E = f$.*

Proof. For $x \in \overline{E}$, let x_n be any sequence in E converging to x. In particular, $\{x_n\}_{n\ge1}$ is Cauchy. Since f is uniformly continuous, $\{f(x_n)\}_{n\ge1}$ is Cauchy in $\mathbb{R}$ (Verify that uniformly continuous function takes Cauchy sequence to Cauchy sequence). Since $\mathbb{R}$ is complete, $f(x_n)$ converges so define $\tilde{f} : \overline{E} \to \mathbb{R}$ by $\tilde{f}(x) = \lim f(x_n)$ for any sequence $\{x_n\} \subset E$ converging to x. First we have to show $\tilde{f}$ is well-defined. Suppose $x_n \to x$ and $y_n \to x$. Let $\epsilon > 0$. We would like to show $|f(x_n) - f(y_n)| < \epsilon$ for n sufficiently large, i.e. $\lim f(x_n) = \lim f(y_n)$. Since f is uniformly continuous, there exists $\delta > 0$ such that $d_X(x, y) < \delta \implies |f(x) - f(y)| < \epsilon$. Since $x_n \to x$ and $y_n \to x$, there exists N such that $d_X(x_n, y_n) < \delta$ for all $n \ge N$. Hence for any ϵ, there exists N such that $|f(x_n) - f(y_n)| < \epsilon$ for $n \ge N$. It follows that $\lim f(x_n) - f(y_n) = 0$, or $\lim f(x_n) = \lim f(y_n)$. Therefore $\tilde{f}$ is well-defined. For $x \in E$, we can choose the constant sequence $x, x, x, \cdots$ converging to x so that $\tilde{f}(x) = \lim f(x) = f(x)$. This shows that $\tilde{f}$ is an extension of f, i.e. $\tilde{f}|_E = f$. The continuity of f follows from definition. The uniqueness follows from the fact that a continuous mapping is determined by its values on a dense subset of its domain (Problem 4.4). ■

With this lemma, the problem becomes quite trivial. We have an extension $\tilde{f} : \overline{E} \to \mathbb{R}$. $\overline{E}$ is compact by Heine Borel so $\tilde{f}(\overline{E})$ is compact and hence bounded. Finally observe that $f(E) \subset \tilde{f}(\overline{E})$. Note that $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x$ is a unbounded uniformly continuous function.
Remark. *Note that the above continuous extension lemma can be generalized by taking the codomain to be any complete metric space Y.*

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4.14

Problem. *Let $I = [0,1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.*

Proof. Consider the continuous function $g(x) = f(x) - x$. $g(0) = f(0)$ and $g(1) = f(1) - 1$. If $f(0) = 0$ or $f(1) = 1$, we are done; otherwise $g(0) > 0 > g(1)$. By the intermediate value theorem, there exists $x \in (0,1)$ such that $g(x) = 0$, or $f(x) = x$. ■

4.20

Problem. *If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by $\rho_E(x) = \inf_{z \in E} d(x,z)$.*

(a) Prove that $\rho_E(x) = 0$ iff $x \in \overline{E}$.

(b) Prove that ρ_E is a uniformly continuous function on X, by showing that

$$|\rho_E(x) - \rho_E(y)| \le d(x,y).$$

Proof. (a) $\rho_E(x) = 0$ iff there exists $\{x_n\}_{n\ge1}$ a sequence in E such that $d(x,x_n) \to 0$ as $n \to \infty$ iff there exists $\{x_n\}_{n\ge1}$ a sequence in E such that $x_n \to x$ iff $x \in \overline{E}$.

(b) For any $z \in E$, $\rho_E(x) \le d(x,z) \le d(x,y) + d(y,z)$. So

$$\rho_E(x) - d(x,y) \le d(y,z)$$

for all $z \in E$. Take the infimum over $z \in E$, we have $\rho_E(x) - d(x,y) \le \rho_E(y)$. So $\rho_E(x) - \rho_E(y) \le d(x,y)$. By symmetry, $|\rho_E(x) - \rho_E(y)| \le d(x,y)$. So ρ_E is uniformly continuous by the first homework problem taking $C = 1$.

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4.21

Problem. *Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p,q) > \delta$ if $p \in K, q \in F$. Show that the conclusion may fail for two disjoint closed sets if neither is compact.*

Proof. Consider the function ρ_F. $\rho_F(x) = 0$ iff $x \in \overline{F} = F$ by the previous problem so ρ_F is positive on $X \setminus F$. In particular, ρ_F is a positive continuous function on K. Since K is compact, $\rho_F(K)$ is compact and so attains a minimum in $(0,\infty)$, i.e. there exists $m \in K$ such that $0 < \rho_F(m) \le \rho_F(k)$ for all $k \in K$. Take $\delta > 0$ to be any number less than $\rho_F(m)$. Then for all $p \in K, q \in F$, $d(p,q) \ge \rho_F(p) \ge \rho_F(m) > \delta$. If neither is compact, we can take subsets of $\mathbb{R}^2$, $K = \{(x,0) : x \in \mathbb{R}\}$ and $F = \{(x,y) : xy = 1\}$. K and F are closed, but $y \to 0$ as x approaches infinity in F. ■