Midterm 2 - Solutions

Problem 1. Let \( a_1 = 1 \) and define \( \{a_n\}_{n \geq 1} \) recursively by
\[
a_{n+1} = \frac{1 + a_n}{2 + a_n}.
\]
Prove that \( \{a_n\} \) is monotone and bounded. Find \( \lim_{n \to \infty} a_n \).

This is Example 2.31, page 44 in textbook.

Problem 2. Prove that the following polynomial
\[
P(x) = x^4 - 4x^3 + x + 1
\]
has at least two real distinct roots (A root is a number \( a \in \mathbb{R} \) satisfying \( P(a) = 0 \)).

Solution. By direct computation \( P(0) = 1 > 0, P(1) = -1 < 0, P(4) = 4^4 - 4 \cdot 4^3 + 4 + 1 = 5 > 0 \). From the intermediate value theorem, it follows that there \( P \) has a zero in the interval \( (0, 1) \) and another one in the interval \( (1, 4) \); since the two intervals are disjoint, the two zeros are different. Therefore \( P \) has at least 2 zeros.

Problem 3. i) Write down the definition of continuity for \( f : D \to \mathbb{R} \) at a point \( x_0 \in D \).

ii) Write down the \( \epsilon - \delta \) criterion for \( f : D \to \mathbb{R} \) at a point \( x_0 \in D \).

iii) Prove that the following function \( f : \mathbb{R} \to \mathbb{R} \),
\[
f(x) = \begin{cases} 
0, & x \in \mathbb{Q} \\
1, & x \in \mathbb{R} \setminus \mathbb{Q}
\end{cases}
\]
is not continuous.

iv) Suppose that the function \( g : \mathbb{R} \to \mathbb{R} \) is continuous and \( g(x) = 2x + 1 \) for all \( x \in \mathbb{R} \setminus \mathbb{Q} \). Prove that \( g(x) = 2x + 1 \) for all \( x \in \mathbb{R} \).

Solution. For i)-iii) see notes or textbook.

iv) All we need to show is that if \( x_0 \in \mathbb{Q} \) then \( g(x_0) = 2x_0 + 1 \). Since \( \mathbb{R} \setminus \mathbb{Q} \) is dense in \( \mathbb{R} \), there exists a sequence \( \{x_n\} \) in \( \mathbb{R} \setminus \mathbb{Q} \) with \( \lim_{n \to \infty} x_n = x_0 \). Since \( g \) is continuous at \( x_0 \), we obtain that \( g(x_0) = \lim_{n \to \infty} g(x_n) = \lim_{n \to \infty} (2x_n + 1) = 2x_0 + 1 \).