HOMEWORK #2 – SOLUTIONS

Solutions to Problem 1.2.1. a. False. Consider the interval \( I = (0.2, 0.3) \). Then there’s no element from \( \mathbb{Z} \) in \( I \), so \( \mathbb{Z} \) is not dense in \( \mathbb{R} \).

b. False. Consider the interval \( I = (-2, -1) \). Then \( I \cap (0, \infty) = \emptyset \).

c. Yes. Let \((a, b)\) be an arbitrary interval in \( \mathbb{R} \). Since \( \mathbb{Q} \) is dense in \( \mathbb{R} \), there exists \( q_0 \in \mathbb{Q} \) such that \( q_0 \in (a, b) \).

- If \( q_0 \notin \mathbb{N} \) then \( q_0 \in \mathbb{Q} \setminus \mathbb{N} \).
- If \( q_0 \in \mathbb{N} \) then \((q_0, q_0 + 1) \cap (a, b)\) contains no integer. Since \( \mathbb{Q} \) is dense in \( \mathbb{R} \), there exists \( q \in \mathbb{Q} \) such that \( q \in (q_0, q_0 + 1) \cap (a, b) \). From the remark above, \( q \) is not an integer, so \( q \in \mathbb{Q} \setminus \mathbb{N} \).

In both cases, we found a number \( q \in \mathbb{Q} \setminus \mathbb{N} \) such that \( q \in (a, b) \). This implies that \( \mathbb{Q} \setminus \mathbb{N} \) is dense in \( \mathbb{R} \).

Solutions to Problem 1.2.2. Let \( m = \inf S \).

Claim: \([m, m + 1]\) contains at least one element in \( S \).

Proof: Suppose that \([m, m + 1]\) doesn’t contain any element in \( S \), then \( m + 1 \leq a \) for all \( a \in S \). In other words, \( m + 1 \) is a lower bound, but this gives a contradiction since \( m \) is the greatest lower bound.

Let \( n \) be the element in \( S \cap [m, m + 1] \). Thus \( n < m + 1 \). Then, for any element \( k < n \), we have

\[
k \leq n - 1 < m.
\]

Since \( m \) is a lower bound, this implies that \( k \notin S \) for all \( k < n \). Therefore, \( n \) is a minimum of \( S \).

Solutions to Problem 1.2.3. Define \( a = \inf S \). Assume first that \( a \in S \). By the definition of the infimum, for any \( b \in S, a \leq b \). Therefore, \( a \) is the minimum of \( S \). Conversely, assume that \( m \) is the minimum of \( S \). Since \( m \in S \), so \( \inf S \leq m \). However, \( m \) is a lower bound of \( S \), so \( m \leq \inf S \). Therefore, we can conclude that \( m = \inf S \).

Solutions to Problem 1.2.5. Suppose for a contradiction that \( a > 0 \). Then, by the Archimedean Property, there exists some \( n_0 \in \mathbb{N} \) s.t. \( \frac{1}{n_0} < a \). But this contradicts the hypothesis \( a \leq \frac{1}{n} \) for every \( n \in \mathbb{N} \).

2nd Method. From the hypothesis on \( a \), we have that \( a \) is a lower bound for the set \( S := \{ \frac{1}{n} : n \in \mathbb{N} \} \). From Ex. 1.2.4(a), \( \inf(S) = 0 \). Since \( \inf(S) \) is the greatest lower bound of \( S \), we deduce that \( a \leq 0 \).

Solutions to Problem 1.2.6. Let \( b \in \mathbb{R} \) be an upper bound for \( S \). Suppose that \( b < a \). Since \( \mathbb{Q} \) is dense in \( \mathbb{R} \), then there exist \( c \in (b, a) \cap \mathbb{Q} \). Thus, \( c \in S \), which implies that \( b \) is not an upper bound of \( S \). This is in contradiction. So \( b \geq a \). This implies that \( \sup S \geq a \). On the other hand \( a \) is an upper bound of \( S \). This implies now that \( \sup S = a \).

Solutions to Problem 1.3.4. From Proposition 1.12, the fact that \( |x - a| < a/2 \) yields \( x > a - a/2 = a/2 \).

Solutions to Problem 1.3.7. By the triangle inequality

\[
|\langle a + b \rangle - b| \leq |a + b| + |b|
\]

\[
|a| \leq |a + b| + |b|
\]

\[
|a| - |b| \leq |a - b|.
\]

By switching between \( a \) and \( b \), we have

\[
|b| - |a| \leq |b - a| = |a - b|.
\]

Thus \( |a - b| \geq |a| - |b| \) and \( |a - b| \geq -(|a| - |b|) \). Therefore,

\[
|a - b| \geq ||a| - |b||.
\]

Solutions to Problem 1.3.9. Note that

\[
a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1}).
\]

Because \( a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1} \geq 0 \), so \( a - b \leq 0 \) implies \( a^n - b^n \leq 0 \) and vice versa.
Problem 1.3.11. Using the Binomial Formula, we can write down:
\[(1 + b)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} b^k \geq \binom{n}{0} b^0 + \binom{n}{1} b^1 = 1 + nb\]
where for the inequality, we kept from the sum (of nonnegative numbers) only the first two terms.

Problem 1.3.14. Since \((a - b)^2\) is always positive, we have
\[
0 \leq (a - b)^2 \\
0 \leq a^2 - 2ab + b^2 \\
2ab \leq a^2 + b^2 \\
ab \leq \frac{1}{2}(a^2 + b^2).
\]