Solution to Problem 2.1.1. a. False: let \( a_n = (-1)^n \). Then \( a_n^2 = 1 \) for all \( n \in \mathbb{N} \), so \( \{a_n^2\} \) converges but \( \{a_n\} \) doesn’t converge.

b. False: let \( a_n = (-1)^n \) and \( b_n = (-1)^{n+1} \). Then \( a_n + b_n = 0 \) for all \( n \in \mathbb{N} \), so \( \{a_n + b_n\} \) converges but \( \{a_n\} \) and \( \{b_n\} \) doesn’t converge.

c. True: since \( \{a_n\} \) converges, so \( \{-a_n\} \) converges. Because \( \{a_n + b_n\} \) and \( \{-a_n\} \) converge, we have \( \{a_n + b_n + (-a_n)\} = \{b_n\} \) converges.

d. False: let \( a_n = (-1)^n \). Then \( |a_n| = 1 \) for all \( n \in \mathbb{N} \), so \( \{|a_n|\} \) converges but \( \{a_n\} \) doesn’t converge.

Solution to Problem 2.1.2. a. Let \( \epsilon > 0 \) be arbitrary. We want to show that:

There exists \( N \in \mathbb{N} \) such that \( \left| \frac{1}{\sqrt{n}} - 0 \right| = \frac{1}{\sqrt{n}} < \epsilon \) for all \( n \geq N \).

By the Archimedean Property, there exists \( N \in \mathbb{N} \) such that \( \frac{1}{N} < \epsilon^2 \). This implies that \( \frac{1}{\sqrt{N}} < \epsilon \). Hence, for any \( n \geq N \), we have

\[
\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \epsilon \quad \text{for all } n \geq N
\]
as we wanted.

b. Let \( \epsilon > 0 \) be arbitrary. We want to show that:

There exists \( N \in \mathbb{N} \) such that \( \left| \frac{1}{n + 5} - 0 \right| = \frac{1}{n + 5} < \epsilon \) for all \( n \geq N \).

By the Archimedean Property, there exists \( N \in \mathbb{N} \) such that \( \frac{1}{N} < \epsilon \). This implies \( \frac{1}{N+5} < \frac{1}{N} < \epsilon \). Hence, for any \( n \geq N \), we have

\[
\frac{1}{n + 5} \leq \frac{1}{N + 5} < \epsilon \quad \text{for all } n \geq N,
\]
as desired.

Solution to Problem 2.1.6. Since \( a_n \to a \), for every \( \epsilon > 0 \) there is an index \( N \in \mathbb{N} \) starting from which \( a_n \in (a - \epsilon, a + \epsilon) \) (look at the comment and picture after the “Definition” on p.26). We are free to choose our epsilon to derive some further properties of \( \{a_n\} \). Here for example, we can choose \( \epsilon = \frac{\alpha}{2} \). So there is \( N \in \mathbb{N} \) s.t. \( a_n \in (\frac{\alpha}{2} - \epsilon, \frac{\alpha}{2} + \epsilon) \) for all \( n \geq N \). Since \( \frac{\alpha}{2} > 0, a_n > 0 \) for all \( n \geq N \).

Solution to Problem 2.1.7 Recall that the inequality stated in the comparison lemma doesn’t have to be true for all \( a_n \) and \( b_n \): we just need to find \( N_0 \in \mathbb{N} \) such that

\[ |b_n - l| \leq |a_n - l| \text{ for all } n \geq N_0. \]

In this case, we take \( N_0 \) to be \( N \) from the problem. It follows that

\[ |b_n - l| = |a_n - l| \text{ for all } n \geq N. \]

Since \( a_n \to l \), by the comparison lemma, we have \( b_n \to l \) as well.

Solution to Problem 2.1.8 Suppose \( \{c_n\} \) converges to \( c \). Since the constant sequence \( \{-c\} \) converges to \( -c \), Theorem 2.10 implies that \( \{c_n - c\} \) converges to \( c - c = 0 \). This proves the first part.

For the second part, suppose that \( \{c_n - c\} \) converges to 0. Again, we notice that since the constant sequence \( \{c\} \) converges to \( c \), we obtain using Theorem 2.10 that \( \{(c_n - c) + c\} \) converges to \( c \). This proves the second part.
Solution to Problem 2.1.12 We will prove this by induction. The base case is true since $|1 - \sqrt{2}| < 2$. Now assume that $|a_n - \sqrt{2}| < \frac{2}{n}$. Then $-\frac{2}{n} < a_n - \sqrt{2} < \frac{2}{n}$. There are two possible choices for $a_{n+1}$.

- **Case 1:** $a_n \geq \sqrt{2}$ and $a_{n+1} = a_n - \frac{1}{n}$. We recall the inequalities that we have in this case:
  
  $$-\frac{2}{n} < a_n - \sqrt{2} < \frac{2}{n}, \quad a_n - \sqrt{2} \geq 0$$

  Therefore, we have that
  
  $$0 \leq a_n - \sqrt{2} < \frac{2}{n}.$$  

  We can find the bounds for $a_{n+1} - \sqrt{2}$ by writing
  
  $$a_{n+1} - \sqrt{2} = a_n - \sqrt{2} - \frac{1}{n}$$

  and note that
  
  $$0 - \frac{1}{n} \leq a_n - \sqrt{2} - \frac{1}{n} < \frac{2}{n} - \frac{1}{n},$$

  $$-\frac{1}{n} \leq a_n - \sqrt{2} - \frac{1}{n} < \frac{1}{n}.$$ 

  Therefore,
  
  $$|a_{n+1} - \sqrt{2}| = |a_n - \sqrt{2} - \frac{1}{n}| < \frac{1}{n} < \frac{2}{n+1}.$$ 

- **Case 2:** $a_n < \sqrt{2}$ and $a_{n+1} = a_n + \frac{1}{n}$. In this case, we have
  
  $$-\frac{2}{n} < a_n - \sqrt{2} \leq 0.$$ 

  Consequently,
  
  $$-\frac{1}{n} \leq a_n - \sqrt{2} + \frac{1}{n} < \frac{1}{n},$$ 

  Therefore,
  
  $$|a_{n+1} - \sqrt{2}| = |a_n - \sqrt{2} + \frac{1}{n}| < \frac{1}{n} < \frac{2}{n+1}.$$ 

  It follows from induction that
  
  $$|a_n - \sqrt{2}| < \frac{2}{n}$$ for all $n \in \mathbb{N}$. 

  Since $\frac{2}{n} \to 0$, so by the comparison lemma, $a_n \to \sqrt{2}$.

Solution to Problem 2.1.14 Note that $\frac{1}{(k+1)(k)} = \frac{1}{k} - \frac{1}{k+1}$ for all $k$. Thus

$$s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$ 

Thus,

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right) = 1 - \lim_{n \to \infty} \frac{1}{n} = 1.$$ 

Solution to Problem 2.1.15 a. Note that

$$n^3 - 4n^2 - 100n = n^3 \left(1 - \frac{4}{n} - \frac{100}{n^2}\right).$$ 

Let $c > 0$ be an arbitrary real number. By Archimedean Property, there exists $N_0 \in \mathbb{N}$ such that $N_0 > c$. By choosing $N = \max\{N_0, 100\}$, we have that $N \geq 100$. Thus, for any $n \geq N$, we have (be careful! you want to make sure that the term inside the parenthesis is not a negative number):

$$n^3 \left(1 - \frac{4}{n} - \frac{100}{n^2}\right) \geq \left(1 - \frac{4}{100} - \frac{100}{1000000}\right) > \left(1 - \frac{2}{10}\right) > N > c$$ for all $n \geq N$.  

Therefore, $\lim_{n \to \infty}[n^3 - 4n^2 - 100n] = \infty$.

b. Note that

$$\sqrt{n} - \frac{1}{n^2} + 4 > \sqrt{n} - 1 + 4 = \sqrt{n} + 3 > \sqrt{n}.$$
Let \( c > 0 \) be an arbitrary real number. By Archimedean Property, there exists \( N \in \mathbb{N} \) such that \( N > c^2 \), or equivalently, \( \sqrt{N} > c \), hence, for any \( n \geq N \),
\[
\sqrt{n} - \frac{1}{n^2} + 4 > \sqrt{n} \geq \sqrt{N} > c \quad \text{for all } n \geq N.
\]
Therefore, \( \lim_{n \to \infty} [\sqrt{n} - \frac{1}{n^2} + 4] = \infty \).

**Solution to Problem 2.1.16 a.** Note that
\[
\sqrt{n} + 1 - \sqrt{n} = (\sqrt{n} + 1 - \sqrt{n}) \cdot \frac{\sqrt{n} + 1 + \sqrt{n}}{\sqrt{n} + 1 + \sqrt{n}} = \frac{1}{\sqrt{n} + 1 + \sqrt{n}}.
\]

Consequently,
\[
\lim_{n \to \infty} [\sqrt{n} + 1 - \sqrt{n}] = \lim_{n \to \infty} \left[ \frac{1}{\sqrt{n} + 1 + \sqrt{n}} \right] = 0.
\]

**b.** From (??), we have
\[
\lim_{n \to \infty} [(\sqrt{n} + 1 - \sqrt{n})/\sqrt{n}] = \lim_{n \to \infty} \left[ \frac{\sqrt{n}}{\sqrt{n} + 1 + \sqrt{n}} \right]
= \lim_{n \to \infty} \left[ \frac{1}{\sqrt{1 + \frac{1}{n} + 1}} \right]
= \frac{1}{\sqrt{2}}.
\]

**c.** Note that \( 4n > n + 1 \) for all \( n \in \mathbb{N} \). Taking a square root on both sides, \( 2\sqrt{n} > \sqrt{n + 1} \). From (??), we have
\[
(\sqrt{n} + 1 - \sqrt{n})n = \frac{n}{\sqrt{n} + 1 + \sqrt{n}} > \frac{n}{2\sqrt{n} + \sqrt{n}} = \frac{1}{3}\sqrt{n}.
\]

Let \( c > 0 \) be an arbitrary real number. By Archimedean Property, there exists \( N \in \mathbb{N} \) such that \( N > 9c^2 \), or equivalently, \( \frac{1}{3}\sqrt{N} > c \). Thus for any \( n \geq N \),
\[
\frac{1}{3}\sqrt{n} \geq \frac{1}{3}\sqrt{N} > c.
\]
Therefore, \( \lim_{n \to \infty} [(\sqrt{n} + 1 - \sqrt{n})n] = \infty \).

**Solution to Problem 2.1.17.** We start by writing the \( \varepsilon - N \) definitions for the two limits:
\[
(*) \lim_{n \to \infty} a_n = +\infty : \forall c > 0, \exists N = N(c) \in \mathbb{N} \text{ s.t. } a_n > c, \forall n \geq N;
\]
\[
(**) \lim_{n \to \infty} \frac{1}{a_n} = 0 : \forall \varepsilon > 0, \exists N = N(\varepsilon) \in \mathbb{N} \text{ s.t. } \left| \frac{1}{a_n} - 0 \right| < \varepsilon, \forall n \geq N.
\]

Since \( a_n > 0 \), we have:
\[
(2) \left| \frac{1}{a_n} - 0 \right| < \varepsilon \iff \frac{1}{a_n} < \varepsilon \iff a_n > \frac{1}{\varepsilon}.
\]

We were asked to show (*) \(\iff\) (**).

\(\Rightarrow\): Assume (*) is true. In order to prove (**), let \( \varepsilon > 0 \). Set \( c := \frac{1}{\varepsilon} \) and apply (*).

So, there is \( N \in \mathbb{N} \) s.t. \( a_n > c \iff a_n > \frac{1}{\varepsilon} \) for all \( n \geq N \). By the equivalence (??), we get \( \left| \frac{1}{a_n} - 0 \right| < \varepsilon \) for all \( n \in \mathbb{N} \).

\(\Leftarrow\): Assume (** is true. In order to prove (*), let \( c > 0 \). Set \( \varepsilon := \frac{1}{c} \) and apply (**).

So, there is \( N \in \mathbb{N} \) s.t. \( \left| \frac{1}{a_n} - 0 \right| < \varepsilon \iff a_n > \frac{1}{\varepsilon} = c \) for all \( n \geq N \) (here we used again (??)). Therefore, we have \( a_n > c \) for all \( n \in \mathbb{N} \).