Solution to Problem 2.4.1. For this problem, let \( \{a_n\} \) be a sequence and \( \{a_{n_k}\} \) be its subsequence.

(a) True: If there exists \( M > 0 \) such that \( |\{a_n\}| < M \) for all \( n \in \mathbb{N} \), then \( |\{a_{n_k}\}| < M \) for all \( k \in \mathbb{N} \).

(b) True: Assume that \( \{a_n\} \) is increasing. This implies that for any \( m, n \in \mathbb{N} \) such that \( m < n \), we have \( a_m < a_n \). Since \( n_k < n_{k+1} \), we have \( a_{n_k} < a_{n_{k+1}} \) for all \( k \in \mathbb{N} \). Therefore, \( \{a_{n_k}\} \) is increasing. The same argument with all inequalities reversed applies when \( \{a_n\} \) is decreasing.

(c) True: Assume that \( a_n \rightarrow a \) as \( n \rightarrow \infty \). This means that for any \( \epsilon > 0 \), there exists \( N_1 \in \mathbb{N} \) such that \( |a_n - a| < \epsilon \) for all \( n \geq N_1 \). Since the index \( n_k \) is increasing as \( k \) increases, there exists \( N \in \mathbb{N} \) such that \( n_k \geq N_1 \) for all \( k \geq N \), so \( |a_{n_k} - a| < \epsilon \) for all \( k \geq N \). This implies that \( a_{n_k} \rightarrow a \) as \( k \rightarrow \infty \).

(d) False: The sequence \( a_n = (-1)^n \) doesn’t converge but it has a subsequence \( a_{2k} = (-1)^{2k} = 1 \) for all \( k \in \mathbb{N} \) which converges.

Solution to Problem 2.4.3.

(a) \( \frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \frac{1}{13}, \frac{1}{16} \).

(b) \( \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10} \).

(c) \( \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25} \).

Solution to Problem 2.4.7. Suppose \( \{a_{n_k}\} \) is a bounded subsequence of \( \{a_n\} \). We need to notice that the subsequence has the same monotonicity direction as \( \{a_n\} \). By MCT, \( \{a_{n_k}\} \) converges to \( \alpha := \sup_{k \in \mathbb{N}} a_{n_k} \) (notice that the supremum is taken only over the terms of the subsequence). Now, for every \( k \in \mathbb{N} \), we have \( n_k \geq k \) and thus \( a_{n_k} \geq a_k \). Since \( a_{n_k} \leq \alpha \) and \( a_k \leq a_{n_k} \), we have \( a_k \leq \alpha \). Therefore, every term \( a_k \) of the sequence is less or equal than \( \alpha \). So \( \{a_n\} \) is bounded above, and since it’s clearly bounded below by \( a_1 \), it is a bounded sequence.

The case when \( \{a_n\} \) is monotonically decreasing can be treated analogously, or you can employ the above argument for the sequence \( \{−a_n\} \).

Solution to Problem 2.4.8. Suppose \( \{a_{n_k}\} \) is a convergent subsequence of \( \{a_n\} \). Then \( \{a_{n_k}\} \) is bounded (any convergent sequence is bounded). By Problem 2.4.7, \( \{a_n\} \) is bounded. So \( \{a_n\} \) is a monotone and bounded sequence. By MCT, \( \{a_n\} \) converges.

Solution to Problem 2.4.9 “⇒”: Suppose that \( \{a_n\} \) is bounded, so there is a number \( M \) such that \( |a_n| \leq M \) for all \( n \). Thus, \( −M \leq a_n \leq M \) for all \( n \). From here, we can choose \( a = −M \) and \( b = M \).

“⇐”: Suppose that there exist \( a \) and \( b \) such that \( a \leq a_n \leq b \) for all \( n \). Then, by letting \( M = \max\{|a|, |b|\} \), then it follows that \( −M \leq a \leq a_n \leq b \leq b \) for all \( n \).

Solution to Problem 2.4.10 “⇒”: Assume that \( \{a_n\} \) does not converge to \( a \), i.e.

\[ \exists \epsilon > 0 \text{ s.t. } \forall N \in \mathbb{N}, \, \exists n \geq N \text{ with } |a_n - a| \geq \epsilon . \]

We’ll apply this property successively. For \( N = 1 \), there exists \( n_1 \geq 1 \) with \( |a_{n_1} - a| \geq \epsilon \). Then, for \( N = n_1 + 1 \) there exists \( n_2 \geq n_1 + 1 \) with \( |a_{n_2} - a| \geq \epsilon \). Inductively (* this means that the statement can be proved using the Induction Principle), for each \( k \in \mathbb{N} \), taking \( N = n_k + 1 \), there exists \( n_{k+1} \geq n_k + 1 \) with \( |a_{n_{k+1}} - a| \geq \epsilon \). So, we have produced an increasing sequence of indices \( \{n_k\} \) (i.e. we have given a subsequence \( \{a_{n_k}\} \) of \( \{a_n\} \)) satisfying \( |a_{n_k} - a| \geq \epsilon \) for all \( k \in \mathbb{N} \).

“⇐”: Assume that there exist \( \epsilon_0 > 0 \) and a subsequence \( \{a_{n_k}\} \) s.t. \( |a_{n_k} - a| \geq \epsilon_0 \) for all \( k \in \mathbb{N} \). Suppose for a contradiction that \( \{a_n\} \) converges to \( a \), i.e.

\[ \forall \epsilon > 0 \text{ , } \exists N_{\epsilon} \in \mathbb{N} \text{ s.t. } |a_n - a| < \epsilon , \forall n \geq N_{\epsilon} . \]

In particular for \( \epsilon = \epsilon_0 \), there is \( N_0 := N_{\epsilon_0} \in \mathbb{N} \text{ s.t. } |a_n - a| < \epsilon_0 \) for all \( n \geq N_0 \). For any \( k \geq N_0 \), we have \( n_k \geq k \geq N_0 \), so \( |a_{n_k} - a| < \epsilon_0 \). This contradicts the assumption. Then it must be that \( \{a_n\} \) does not converge to \( a \).
Solution to Problem 3.1.1. a. False. Let $f(x) = 1$ if $x \geq 1$ and $f(x) = -1$ otherwise, and $g(x) = -1$ if $x \geq 1$ and $g(x) = 1$ otherwise. Then, $f + g \equiv 0$ is continuous but $f$ and $g$ are not continuous.

b. False. Let $f(x)$ be the function in a. Then, $f^2 \equiv 1$ is continuous.

c. True. Since $f + g$ and $g$ are continuous, so $f = (f + g) - g$ is continuous.

d. True. Let $x \in \mathbb{N}$ and $\{x_n\}$ be a sequence in $\mathbb{N}$ such that $\lim_{n \to \infty} x_n = x$. Then, by choosing $\epsilon = \frac{1}{2}$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \frac{1}{2}$ for all $n \geq N$. Since $x_n$ and $x$ are natural numbers, this implies that $x_n = x$ for all $n \geq N$. Therefore, for any $\epsilon > 0$, $|f(x_n) - f(x)| = 0 < \epsilon$ for all $n \geq N$. From this, we conclude that $\lim_{n \to \infty} f(x_n) = f(x)$.

Solution to Problem 3.1.3. We can see that $f(x)$ is not continuous at $x = 0$. To show this, we define two sequences $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n}$. Then $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 0$ but $\lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} \frac{1}{n} + 1 = 1$ and $\lim_{n \to \infty} f(b_n) = \lim_{n \to \infty} \frac{1}{n^2} = 0$. Therefore, $f(x)$ is not continuous at $x = 0$.

We will show that $f$ is continuous at any other points $x \neq 0$. First, we assume that $x > 0$. Let $\{x_n\}$ be a sequence such that $\lim_{n \to \infty} x_n = x$. Then, by choosing $\epsilon = \frac{x}{2}$, there is $N \in \mathbb{N}$ such that $x_n > x - \frac{x}{2} = \frac{x}{2} > 0$ for all $n \geq N$. Therefore, $f(x_n) = x_n + 1$ for $n \geq N$. Defining another sequence $g_n = x_n + 1$, we have that

$$|f(x_n) - (x + 1)| = |g(x_n) - (x + 1)| \quad \text{for } n \geq N.$$

By comparison lemma, we have that $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = x + 1$.

Solution to Problem 3.1.4. Denote by $f_A$, $f_B$ the restriction of $f$ to $A$ and $B$, respectively ($f_A(x) = f(x)$ for every $x \in A$ and $f_B(y) = f(y)$ for every $y \in B$).

Notice that $A \cap B = \{x_0\}$ and that $A \cup B = D$.

“$\Rightarrow$”: Assume that $f$ is continuous at $x_0$. Let $\{x_n\}$ be a sequence in $A$ converging to $x_0$. Then $x_n \in D$ and $x_n \to x_0$. Then $f_A(x_0) = f(x_0)$, $f_A(x_n) = f(x_n)$ for all $n \in \mathbb{N}$ and since $f$ is continuous, $\{f(x_n)\}$ converges to $f(x_0)$. It follows that $f_A(x_n) \to f_A(x_0)$. This shows that $f_A$ is continuous at $x_0$.

In a similar way (start by letting a sequence $\{y_n\}$ in $B$ converge to $x_0$), we get the continuity of $f_B$ at $x_0$.

“$\Leftarrow$”: Assume now that both $f_A$ and $f_B$ are continuous at $x_0$. Let $\{x_n\}$ be a sequence in $D$ converging to $x_0$. We have:

$$f(x_n) = \begin{cases} f_A(x_n), & \text{if } x_n \in A \\ f_B(x_n), & \text{if } x_n \in B \end{cases}$$

(notice that the two branches agree on $A \cap B$).

We then define the following two sequences:

$$y_n = \begin{cases} x_n, & \text{if } x_n \in A \\ x_0, & \text{if } x_n \notin A \end{cases}, \quad z_n = \begin{cases} x_0, & \text{if } x_n \notin B \\ x_n, & \text{if } x_n \in B \end{cases}$$

Notice that $\{y_n\}$ is a sequence in $A$ and $|y_n - x_0|$ is either $|x_n - x_0|$ or 0; in any case it’s definitely less or equal than $|x_n - x_0|$, for every $n \in \mathbb{N}$. By the Comparison Lemma, we get that $\{y_n\}$ converges to $x_0$. Since $f_A$ is continuous, $\{f_A(y_n)\}$ converges to $f_A(x_0) = f(x_0)$. Similarly, we have that $\{z_n\}$ is a sequence in $B$ converging to $x_0$ and using the continuity of $f_B$, we get $\{f_B(z_n)\}$ converges to $f_B(x_0) = f(x_0)$. We have

$$|f(x_n) - f(x_0)| = \begin{cases} |f_A(y_n) - f_A(x_0)|, & \text{if } x_n \in A \\ |f_B(z_n) - f_B(x_0)|, & \text{if } x_n \in B \end{cases}$$

In both branches, the corresponding quantity is less or equal than $|f_A(y_n) - f_A(x_0)| + |f_B(z_n) - f_B(x_0)|$. So we have

$$|f(x_n) - f(x_0)| \leq |f_A(y_n) - f_A(x_0)| + |f_B(z_n) - f_B(x_0)| \quad \forall n \in \mathbb{N}$$

The sequence in the right hand side above converges to 0. By the Comparison Lemma, we get that $f(x_n) \to f(x_0)$.

So for any sequence $\{x_n\}$ in $D$ converging to $x_0$ we have that $\{f(x_n)\}$ converges to $f(x_0)$, i.e. $f$ is continuous.

Solution to Problem 3.1.5. From the Problem 3.1.4, let $D = \mathbb{R}$, $A = \{x \in \mathbb{R} | x \geq 0\}$ and $B = \{x \in \mathbb{R} | x \leq 0\}$. On $A$, $f(x) = x^2$ which is a polynomial, so $f$ is continuous on $A$. Similarly, $f$ is continuous on $B$. Using Problem 3.1.4, we can conclude that $f$ is continuous on $A \cup B = \mathbb{R}$.

Solution to Problem 3.1.6. We’ll show that $g$ is continuous only at $x_0 = 0$. 
First, let \( \{x_n\} \) be a sequence in \( \mathbb{R} \) converging to 0. For each \( n \in \mathbb{N} \) we either have \( g(x_n) = x_n^2 \) or \( g(x_n) = -x_n^2 \); in any case we have \( |g(x_n)| = x_n^2 \). Since \( \{x_n\} \) converges to 0, by the Product Rule, \( \{x_n^2\} \) converges to 0, and therefore by the Comparison Lemma \( \{g(x_n)\} \) converges to 0. Therefore \( g \) is continuous at 0.

Secondly, say \( x_0 \neq 0 \). Since \( \mathbb{Q} \) is sequentially dense in \( \mathbb{R} \), there is a sequence \( \{u_n\} \) in \( \mathbb{Q} \) convergent to \( x_0 \). Then \( g(u_n) = u_n^2 \to x_0^2 \) as \( n \to \infty \). We also have that \( \mathbb{R} \setminus \mathbb{Q} \) is sequentially dense in \( \mathbb{R} \), so there is a sequence \( \{v_n\} \) in \( \mathbb{R} \setminus \mathbb{Q} \) converging to \( x_0 \). Therefore \( g(v_n) = -v_n^2 \to -x_0^2 \). So we produced two sequences, both converging to \( x_0 \) but for which we have

\[
\lim_{n \to \infty} g(u_n) = x_0^2 = -x_0^2 = \lim_{n \to \infty} g(v_n).
\]

It follows that \( g \) is not continuous at \( x_0 \).

**Solution to Problem 3.1.7.** Define a sequence \( a_n = 1 - \frac{1}{n} \) for all \( n \in \mathbb{N} \). Then \( a_n \in [0, 1) \) and so \( f(a_n) \geq 2 \). In other words, \( f(a_n) \in [2, \infty) \) for all \( n \in \mathbb{N} \). Since \( a_n = 1 - \frac{1}{n} \to 1 \), by the continuity of \( f \), we have \( f(a_n) \to f(1) \). Since \( [2, \infty) \) is closed, it follows that \( f(1) \in [2, \infty) \). In other words, \( f(1) \geq 2 \).