HOMEWORK #9 – SOLUTIONS

Solution to Problem 4.1.1. (a) False. Let \( f(x) = |x| \). We see that \( f \) is continuous but \( \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = -1 \) and \( \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = 1 \), so \( f \) is not differentiable at 0.
(b) True. This is Proposition 4.5.
(c) False. Let \( f(x) = |x| \). Then, \( f(x) \) is not differentiable while \( f(x) = x^2 \) is differentiable.

Solution to Problem 4.1.2. We compute \( f'(x) = 3x^2 + 2 \). The equation of the tangent line at \( x_0 = 2 \) is \( y = f(2) + f'(2)(x - 2) = 13 + 14(x - 2) = 14x - 15 \).

Solution to Problem 4.1.3. First, We will show continuity. Let \( x_n \to 0 \). Then,
\[
\lim_{x \to 0^-} f(x_n) = \lim_{x \to 0^-} (m_1 x + 4) = 4 \quad \text{and} \quad \lim_{x \to 0^+} f(x_n) = \lim_{x \to 0^+} (m_2 x + 4) = 4.
\]
Thus, \( \lim_{x \to 0^-} f(x_n) = \lim_{x \to 0^+} f(x_n) \), so \( f \) is continuous at 0.

Now we compute
\[
\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{(m_1 x + 4) - 4}{x} = m_1 \quad \text{and} \quad \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{(m_2 x + 4) - 4}{x} = m_2.
\]
Since we assumed that \( m_1 \neq m_2 \), so the limits are not equal. Consequently, \( f \) is not differentiable at 0.

Solution to Problem 4.1.4. (a) Note that
\[
\frac{\sqrt{x+1} - \sqrt{2}}{x-1} = \frac{\sqrt{x+1} + \sqrt{2}}{x-1} = \frac{\sqrt{x+1} - \sqrt{2}}{x-1} = \frac{x-1}{(\sqrt{x+1} + \sqrt{2})(x-1)} = \frac{1}{\sqrt{x+1} + \sqrt{2}}.
\]
Therefore,
\[
\lim_{x \to 1} \frac{\sqrt{x+1} - \sqrt{2}}{x-1} = \lim_{x \to 1} \frac{1}{\sqrt{x+1} + \sqrt{2}} = \frac{1}{2\sqrt{2}}.
\]
(b) Note that \( f(1) = 1 + 2 = 3 \), and that
\[
\frac{x^3 + 2x - (1+2)}{x-1} = \frac{(x^3-1)+2(x-1)}{x-1} = \frac{(x-1)(x^2+x+3)}{x-1} = x^2+x+3.
\]
Therefore,
\[
\lim_{x \to 1} \frac{x^3 + 2x - (1+2)}{x-1} = \lim_{x \to 1} (x^2+x+3) = 5.
\]
(c) Computing the difference quotient,
\[
\frac{1+x^2 - \frac{1}{2}}{x-1} = \frac{2-(1+x^2)}{2(1+x^2)} = \frac{1-x^2}{2(1+x^2)(x-1)} = \frac{-x+1}{2(1+x^2)},
\]
we have that
\[
\lim_{x \to 1} \frac{1+x^2 - \frac{1}{2}}{x-1} = \lim_{x \to 1} \frac{-x+1}{2(1+x^2)} = -\frac{1}{2}.
\]

Solution to Problem 4.1.8. We notice that: (a) \( f \) is differentiable at every point \( x_0 < 0 \) (since \( f \) is a constant function on \((-\infty, 0])\) having \( f'(x_0) = 0 \); and (b) \( f \) is differentiable at every point \( x_0 > 0 \) (since \( f \) is a polynomial function on \((0, \infty)\)) having \( f'(x_0) = nx_0^{n-1} \).

We are left to study the differentiability at \( x_0 = 0 \). We immediately check that \( f \) is continuous at 0:
\[
\lim_{x \to 0^-} 0 = \lim_{x \to 0^+} x^n = f(0) = 0.
\]
Then, looking at the difference quotient, we have
\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{0}{x} = 0, \quad \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^n}{x} = \lim_{x \to 0} x^{n-1} = 0
\]
Thus, the limit \( \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \) exists and is equal to 0, i.e. \( f \) is differentiable at 0 and \( f'(0) = 0 \).

**Solution to Problem 4.1.9.** Plugging in \( x = 0 \) in the set of inequalities leads us to \( f(0) = 0 \). From \( |f(x)| \leq x^2 \) for all \( x \in \mathbb{R} \), by the Comparison Lemma, we deduce that \( \lim_{x \to 0} f(x) = 0 \). So \( f \) is continuous at 0. Then, we have
\[
|\frac{f(x) - f(0)}{x - 0}| = \frac{|f(x)|}{|x|} \leq \frac{x^2}{|x|} = |x|
\]
for all \( x \neq 0 \). It follows that \( \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0 \), i.e. \( f \) is differentiable at 0 and \( f'(0) = 0 \).

**Solution to Problem 4.1.10.** The idea is to derive two equations in \( a, b \). For \( g \) to be differentiable at 1, it needs at least to be continuous at 1, so we must have
\[
\lim_{x \to 1} g(x) = g(1) \quad \Leftrightarrow \quad \lim_{x \to 1} (a + bx) = 3 \quad \Leftrightarrow \quad a + b = 3
\]
Notice that this condition (namely \( a + b = 3 \)) is also sufficient for continuity of \( g \) at 1.

Secondly, if \( a, b \in \mathbb{R} \) are s.t. \( g \) is continuous at 1, then \( g \) is differentiable at 1 if and only if
\[
\lim_{x \to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x \to 1} \frac{3x^2 - 3}{x - 1} = \lim_{x \to 1} \frac{a + bx - 3}{x - 1}
\]
\[
\Leftrightarrow \quad \lim_{x \to 1} \frac{3(x - 1)(x + 1)}{x - 1} = \lim_{x \to 1} \frac{a + bx - a - b}{x - 1}
\]
\[
\Leftrightarrow \quad \lim_{x \to 1} 3(x + 1) = \lim_{x \to 1} \frac{b(x - 1)}{x - 1} \quad \Leftrightarrow \quad 6 = b
\]
So \( g \) is differentiable at 1 if and only if \( b = 6 \) and \( a = 3 - b = -3 \).

**Solution to Problem 4.1.14.** We can write
\[
f(x_0 + h) - f(x_0 - h) = \frac{f(x_0 + h) - f(x_0) - [f(x_0 - h) - f(x_0)]}{h} = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{f(x_0 - h) - f(x_0)}{h}
\]
\[
= \frac{f(x_0 + h) - f(x_0)}{h} + \frac{f(x_0 + (-h)) - f(x_0)}{-h}
\]
Since \( f \) is differentiable at \( x_0 \), both limits \( \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \) and \( \lim_{h \to 0} \frac{f(x_0 + (-h)) - f(x_0)}{-h} \) exist and are equal with \( f'(x_0) \). Therefore
\[
\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{h} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} + \lim_{h \to 0} \frac{f(x_0 + (-h)) - f(x_0)}{-h} = 2f'(x_0)
\]

**Solution to Problem 4.1.16.** We have
\[
\frac{f(x^2) - f(0)}{x} = x \cdot \frac{f(x^2) - f(0)}{x^2} \quad , \quad \forall x \neq 0
\]
Since \( f \) is differentiable at \( x_0 \), the limit \( \lim_{x \to x_0} \frac{f(x^2) - f(0)}{x^2} \) exists and is a real number. Clearly \( \lim_{x \to 0} x = 0 \). Thus
\[
\lim_{x \to 0} \frac{f(x^2) - f(0)}{x} = \lim_{x \to 0} x \cdot \lim_{x \to 0} \frac{f(x^2) - f(0)}{x^2} = 0 \cdot f'(0) = 0
\]

**Solution to Problem 4.1.19.** By differentiating the given relation, we obtain
\[
0 + 1 + 2x + \ldots + nx^{n-1} = \frac{-(n+1)x^n(1-x) - (1-x^{n+1})(-1)}{(1-x)^2}
\]
which is
\[ 1 + 2x + \ldots + nx^{n-1} = \frac{1 - (n + 1)x^n + nx^{n+1}}{(1-x)^2} \]

Multiplying the last equation by \( x \) and then adding 1 to both sides, gives us
\[ 1 + x + 2x^2 + \ldots + nx^n = \frac{(1-x)^2 + x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2} = \frac{1 - x + x^2 - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2} \]

We can now differentiate the last equation in order to obtain
\[ 1 + 2^2x + \ldots + n^2x^{n-1} \]
\[ = \frac{[-1 + 2x - (n+1)^2x^n + n(n+2)x^{n+1}](1-x)^2 - [1 - x + x^2 - (n+1)x^{n+1} + nx^{n+2}](2)(1-x)}{(1-x)^4} \]
\[ = \frac{[-1 + 2x - (n+1)^2x^n + n(n+2)x^{n+1}](1-x) + 2[1 - x + x^2 - (n+1)x^{n+1} + nx^{n+2}]}{(1-x)^3} \]
\[ = \frac{1 + x - (n+1)^2x^n + (2n^2 + 2n - 1)x^{n+1} + n^2x^{n+2}}{(1-x)^3} \]

**Solution to Problem 4.2.2.** We notice that \( f \) is continuous and strictly decreasing on \((0, \infty)\) (hence \( f \) is one-to-one). Also, we have \( \lim_{x \to 0} f(x) = 1 \) and \( \lim_{x \to \infty} f(x) = 0 \). So \( \lim(f) = (0, 1) \). The point \( y_0 := \sqrt[3]{5} \) belongs to \((0, 1)\) and we see that \( f(2) = y_0 \). Therefore \( f^{-1}(y_0) = 2 \) and applying the derivative of the inverse function formula, we get
\[ (f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{f'(2)} \]

To compute \( f'(2) \) we use the chain rule for the functions \( y \to y^{-1/2} \) and \( x \to 1 + x^2 \) (which is the inner function):
\[ f'(x) = (-1/2) \cdot (1 + x^2)^{-3/2} \cdot 2x \]
which gives \( f'(2) = -5^{-3/2} \cdot 2 = -2 / 5\sqrt{5} \) and consequently \( (f^{-1})'(y_0) = -2 / 5\sqrt{5} \).

**Solution to Problem 4.2.3.** (i) We have to find the inverse function of \( f : (0, \infty) \to \mathbb{R} \), \( f(x) = \frac{1}{x} \). In the equation \( f(x) = y \) we solve for \( x > 0 \), provided that \( y > 0 \). Thus: \( \frac{1}{x} = y \iff \frac{1}{x^2} = \frac{1}{y} \iff 1/x = \sqrt[3]{y} \iff x = 1/\sqrt[3]{y} \). This implies that \( f^{-1}(y) = 1/\sqrt[3]{y} \) for all \( y > 0 \).

Remark: One could have simply verified that the formula given for \( f^{-1}(y) \) verifies \( f(f^{-1}(y)) = y \) for all \( y > 0 \) and \( f^{-1}(f(x)) = x \) for all \( x > 0 \).

(ii) We need to compute \( (f^{-1})'(y) \) in two ways. First, we can directly determine \( (f^{-1})'(y) = (y^{-1/2})' = (-1/2)y^{-3/2} \) for all \( y > 0 \). Secondly, we use the formula for the derivative of the inverse function:
\[ (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f'(1/\sqrt[3]{y})} = \frac{1}{f'(y^{-1/2})} \]

Since \( f'(x) = -2x^{-3} \), we have that \( f'(y^{-1/2}) = (-2)(y^{-1/2})^{-3} = (-2)y^{3/2} \) which gives \( (f^{-1})'(y) = \frac{1}{(-2)y^{3/2}} \).

**Solution to Problem 4.2.6.** Let \( x > 0 \) and for \( h \neq 0 \) with \( |h| \) small enough, we have
\[ \frac{g(x + h) - g(x)}{h} = c \cdot \frac{f(cx + ch) - f(cx)}{ch} \]

Since \( f \) is differentiable, we get that \( g \) is differentiable at \( x \) and
\[ g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = c \cdot \lim_{h \to 0} \frac{f(cx + ch) - f(cx)}{ch} = c \cdot f'(cx) \]

**Solution to Problem 4.2.9.** We can differentiate (w.r.t \( x \)) the relation \( f(x) = -f((-1)x) \) to get (see the above problem) \( f'(x) = -(-1)f'((-1)x) \iff f'(x) = f'(-x) \).