Solution to Homework 1
Math 140B

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Disclaimer: The solution may contain errors or typos so use at your own risk.

5.1

Problem. *Let f be defined for all real x, and suppose that*

$$|f(x) - f(y)| \leq (x-y)^2$$

for all real x and y. Prove that f is constant.

Proof. For $x \neq y$, $\frac{|f(x)-f(y)|}{x-y} \leq x - y$. Taking the limit as $y \to x$, $|f'(x)| \leq 0$ implies $f'(x) = 0$ for all x. Hence f is constant. ■

5.2

Problem. *Suppose $f'(x) > 0$ in (a,b). Prove that f is strictly increasing in (a,b), and let g be its inverse function. Prove that g is differentiable, and that*

$$g'(f(x)) = \frac{1}{f'(x)} \qquad a < x < b.$$

Proof. Take $a < x < y < b$. $f(y) - f(x) = f'(\xi)(y-x)$ for some $\xi \in (x,y)$. Since $f'(\xi) > 0$, $f(y) > f(x)$ so f is strictly increasing in (a,b). Then $f((a,b)) = (c,d)$ for some $c,d \in \mathbb{R}$. For $y \in (c,d)$, let $x = g(y)$ so $f(x) = y$. For $k > 0$, there exists $h > 0$ such that $h = g(y+k) - x$ since f is strictly increasing. Observe that $k \to 0$ iff $h \to 0$. So as $k \to 0$,

$$\frac{g(y+k)-g(y)}{k} = \frac{(x+h)-x}{f(x+h)-f(x)} = \left(\frac{f(x+h)-f(x)}{h}\right)^{-1} \to f'(x)^{-1}$$

since $f'(x) > 0$. Hence g is differentiable and $g'(f(x)) = f'(x)^{-1}$. ■

5.4

Problem. *If*

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, \cdots, C_n$ are real constants, prove that the equation $C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$ has at least one real root between 0 and 1.

Proof. Consider the polynomial $p(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1}$. Note that by assumption $p(1) = 0$ and $p(0) = 0$. By mean value theorem, there exists $x \in (0,1)$ such that $p'(x) = C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$. ■

5.5

Problem. *Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to \infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to \infty$.*

Proof. By mean value theorem $g(x) = f(x+1) - f(x) = f'(\xi)$ for some $\xi \in (x, x+1)$. As $x \to \infty$, $\xi \to \infty$ so $g(x) = f'(\xi) \to 0$. ■

5.6

Problem. *Suppose f is continuous for $x \geq 0$; $f'(x)$ exists for $x > 0$; $f(0) = 0$; f' is monotonically increasing. Put $g(x) = \frac{f(x)}{x}$, $x > 0$ and prove that g is monotonically increasing.*

Proof. It suffices to show $g'(x) = \frac{xf'(x) - f(x)}{x^2} \geq 0$ or equivalently $xf'(x) \geq f(x)$ for all $x > 0$. By mean value theorem, $f(x) = f(x) - f(0) = xf'(\xi)$ for some $\xi \in (0, x)$. Since f' is monotonically increasing and $x > 0$, $xf'(x) \geq xf'(\xi) = f(x)$, which is what we need to show. ■

5.9

Problem. *Let f be a continuous real function on $\mathbb{R}$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follows that $f'(0)$ exists?*

Proof.

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} f'(x) = 3$$

by L'Hospital's rule and continuity of f. ■

5.22

Problem. *Suppose f is a real function on $(-\infty, \infty)$.*

(a) *If f is differentiable and $f'(t) \neq 1$ for every $t \in \mathbb{R}$, prove that f has at most one fixed point.*

(b) *Show that the function f defined by $f(t) = t + (1+e^t)^{-1}$ has no fixed point, although $0 < f'(t) < 1$ for all real t.*

(c) *However, if there exists a constant $A < 1$ such that $|f'(t)| \leq 1$ for all $t \in \mathbb{R}$, prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is arbitrary and $x_{n+1} = f(x_n)$, $n = 1, 2, 3, \cdots$.*

(d) *Show that the process described in (c) can be visualized by the zig-zag path*

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to \cdots.$$

Proof. (a) Suppose $x \neq y$ are fixed points of f. Then by mean value theorem $x - y = f(x) - f(y) = f'(t)(x-y)$, implying $f'(t) = 1$, a contradiction.

(b) Suppose t is a fixed point of f, i.e. $t = f(t) = t + (1+e^t)^{-1}$. Then $(1+e^t)^{-1} = 0$, which is impossible.

(c) This is essentially the Banach fixed point theorem (Theorem 9.23 in Rudin) and idea of proof is identical. We will show the sequence $\{x_n\}$ is Cauchy. First observe that by mean value theorem $|x_3 - x_2| = |f(x_2) - f(x_1)| = |f'(\xi)||x_2 - x_1| \leq A|x_2 - x_1|$ for some ξ between x_1 and x_2. By induction $|x_{n+1} - x_n| \leq A^{n-1}|x_2 - x_1|$. Suppose $n > m > N$. Then

$$\begin{aligned}|x_n - x_m| &\leq |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m| \\ &\leq (A^{n-2} + \cdots + A^{m-1})|x_2 - x_1| \\ &\leq A^{m-1}(1-A)^{-1}|x_2 - x_1|.\end{aligned}$$

Since $0 \leq A < 1$, $A^N \to 0$ as $N \to \infty$. So $|x_n - x_m| \to 0$ as $n, m \to \infty$. Hence $\{x_n\}$ is Cauchy. Since $\mathbb{R}$ is complete, $x = \lim x_n$ exists. By continuity of f, $f(x) = \lim f(x_n) = \lim x_{n+1} = x$ so x is a fixed point.

Remark. *By part (a), the above fixed point is unique. The function in part (b) fails to have fixed point. The key observation is that $f'(t) \to 1$ as $t \to \infty$ whereas in part (c) the derivative is bounded away from 1 by a constant.*

(d) Consider the $\{(x_n, f(x_n))\}_{n=1}^\infty \cup \{(x_n, x_n)\}_{n=2}^\infty$ as a subset of $\mathbb{R}^2$. The goal is to eventually land on the line $y = x$.

∎

5.23

Problem. *The function f defined by*

$$f(x) = \frac{x^3+1}{3}$$

has three fixed points, say α, β, γ, where $-2 < \alpha < -1$, $0 < \beta < 1$, $1 < \gamma < 2$. For arbitrary chosen x_1, define $\{x_n\}$ by setting $x_{n+1} = f(x_n)$.

(a) If $x_1 < \alpha$, prove that $x_n \to -\infty$ as $n \to \infty$.

(b) If $\alpha < x < \gamma$, prove that $x_n \to \beta$ as $n \to \infty$.

(c) If $\gamma < x_1$, prove that $x_n \to \infty$ as $n \to \infty$.

Thus β can be located by this method, but α and γ cannot.

Proof. (a) If $x < \alpha$, then $f(x) - f(\alpha) = f'(\xi)(x-\alpha)$ for some $\xi \in (x, \alpha)$. $f'(x) = x^2$ so $\xi < \alpha < -1$ implies $f'(x) > 1$. So $f(x) - f(\alpha) = f(x) - \alpha < x - \alpha$ so $f(x) < x$. It follows that $\{x_n\}$ is a strictly decreasing sequence. If x_n is bounded below, then x_n converges, say to $x \in \mathbb{R}$. So x must satisfy $f(x) = x$ (Why?), i.e. x is a fixed point. But there is no fixed point on $(-\infty, x)$ for $x < \alpha$. Hence $x_n \to -\infty$.

(b) Let $\alpha < x < \gamma$. If $x_1 = \beta$, then $\{x_n\}$ is the constant sequence $\beta, \beta, \cdots$. So assume $x_1 \neq \beta$. The goal is to show $|f(x) - \beta| < |x - \beta|$ and $f(x)$ and x are on the same side of β, i.e. $x < f(x) < f(f(x)) < \cdots < \beta$ or $\beta < \cdots < f(f(x)) < f(x) < x$. In other words, we would like to show if $\alpha < x_1 < \beta$, f induces a sequence $\{x_n\}$ increasing monotonically to β; if $\gamma > x_1 > \beta$, f induces a sequence $\{x_n\}$ decreasing monotonically to β. The obvious attempt is $f(x) - \beta = f(x) - f(\beta) = f'(\xi)(x - \beta)$ for some ξ between x and β. If $0 < f'(\xi) < 1$, we are done; but $f'(\xi) = \xi^2$ and ξ, in between x and β, could be greater than 1 or less than -1. This means that the mean value theorem is not strong enough in this case. Let $g(x) = \begin{cases} \frac{f(x)-f(\beta)}{x-\beta} & x \neq \beta \\ f'(\beta) & x = \beta \end{cases}$. We would like tighter estimates on $g(x)$. Note that $g(x) = \frac{1}{x-\beta}\left(\frac{x^3-\beta^3}{3}\right) = \frac{x^2+\beta x+\beta^2}{3} = \frac{1}{3}(x+\frac{\beta}{2})^2 + \frac{\beta^2}{4}$ is a parabola. So the minimum of g is given by $\frac{\beta^2}{4}$ at $x = -\frac{\beta}{2}$. Also note that $g(\alpha) = g(\gamma) = 1$. It follows that $0 < \frac{\beta^2}{4} < g(x) < 1$ for $x \in (\alpha, \gamma)$. Then it follows that for $x \neq \beta$, $f(x) - \beta = g(x)(x - \beta)$, where $g(x) \in (0,1)$. So $\{x_n\}$ is monotonic and converges to a fixed point (same reason as part (a)) between β and x_1. Since the only fixed point in this interval is β, $x_n \to \beta$.

(c) Similar to part (a).

■