MATH 140B - Winter 2019

Partial solutions to Homework 2

Exercise 5.11 Since both nominator and denominator go to zero as $h \to 0$, we can apply l'Hôpital's rule to conclude that

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2} &= \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}\\ &= \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{2h}+\lim_{h\to 0}\frac{f'(x)-f'(x-h)}{2h}\\ &= f''(x)\end{aligned}$$

since f is assumed twice differentiable.
Letting

$$f(x)=\begin{cases}1, & \text{if } x>0,\\ 0, & \text{if } x=0,\\ -1, & \text{if } x<0,\end{cases}$$

it is easy to check that the above limit exists and equals zero, whereas f is not even continuous at zero.

Exercise 5.12 One can easily calculate that

$$f'(x)=\begin{cases}3x^2, & \text{if } x\geq 0,\\ -3x^2, & \text{if } x<0,\end{cases}$$

and

$$f''(x)=\begin{cases}6x, & \text{if } x\geq 0,\\ -6x, & \text{if } x<0.\end{cases}$$

$f^{(3)}(0)$ does not exist as the left third derivative at zero is -6, and the right one is 6.

Exercise 5.14 From the definition, it follows as in Exercise 4.23 from last quarter that f is convex on (a,b) if and only if

$$\frac{f(z)-f(x)}{z-x}\leq\frac{f(y)-f(z)}{y-z}$$

for all $a<x<z<y<b$. From this the solution follows easily by applying the mean value theorem.
The claim about f'' follows immediately since a differentiable function is monotonically increasing if and only if its derivative is nonnegative.

Exercise 5.15 Let $x\in(a,\infty), h>0$. Following the hint and taking $\alpha=x, \beta=x+2h$ in Taylor's theorem, we get

$$f'(x)=\frac{1}{2h}(f(x+2h)-f(x))-hf''(\xi)$$

for some $\xi\in(x,x+2h)$. Taking absolute values and using the triangle inequality, we get

$$M_1\leq\frac{M_0}{h}+hM_2.$$

Plugging in $h = \sqrt{\frac{M_0}{M_2}}$ gives the desired result.
For vector valued functions, the same inequality holds. One can proceed as follows. Let $\varepsilon > 0$. Take x_0 such that $|\mathbf{f}'(x_0)| \geq M_1 - \varepsilon$, let $\mathbf{v} = \frac{\mathbf{f}'(x_0)}{|\mathbf{f}'(x_0)|}$ and consider

$$\varphi(t) = \mathbf{v} \cdot \mathbf{f}(t).$$

Denoting by M_i^φ the corresponding constants for the (real-valued) function φ, one easily checks that $M_0^\varphi \leq M_0, M_2^\varphi \leq M_2$, and $M_1^\varphi \geq |\varphi'(x_0)| \geq M_1 - \varepsilon$. From this the result immediately follows.

Exercise 5.16 The assumptions mean that, in the terminology of exercise 5.15, $M_2 < \infty$ and $M_0 \to 0$ as $a \to \infty$. Hence $M_1 \to 0$ as $a \to \infty$, i.e. $f'(x) \to 0$ as $x \to \infty$.

Extra Problem 1. Applying Taylor's theorem with $n = 3$ to $\cos(x)$, we get

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{\sin(\xi)}{3!} x^3$$

for some $\xi \in (0, \frac{\pi}{2})$. As sin is positive on this interval, we get the left inequality. For $n = 5$ we similarly get

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{\sin(\eta)}{5!} x^5,$$

for some $\eta \in (0, \frac{\pi}{2})$, and thus the right inequality.

Exercise 6.1 Since f is only nonzero at x_0, it is easy to see that

$$L(P, f, \alpha) = 0$$

for any partition P, and

$$U(P, f, \alpha) = \Delta(\alpha_i)$$

where $\alpha(i)$ is the interval in the partition P containing x_0. Since α is continuous at x_0, we get $\inf U(P, f, \alpha) = 0$. In other words $f \in \mathfrak{R}(\alpha)$ and $\int f d\alpha = 0$.

Exercise 6.4 Fix $a < b$. Since both the rational and irrational numbers are dense, we see that

$$L(P, f) = 0,$$

and

$$U(P, f) = b - a,$$

for any partition P of $[a, b]$. Hence $\inf U(P, f) = b - a > 0 = \sup L(P, f)$, and so $f \notin \mathfrak{R}$ on $[a, b]$.