Solution to Homework 3
Math 140B

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Disclaimer: The solution may contain errors or typos so use at your own risk.

6.2

Problem. *Suppose $f \geq 0$, f is continuous on $[a,b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.*

Proof. Suppose on the contrary that there exists $x \in [a,b]$ such that $f(x) > 0$. By continuity, there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{f(x)}{2}$ for all $y \in (x-\delta, x+\delta)$. Refine δ so that $(x-\delta, x+\delta) \subset [a,b]$. Then $\inf_{y \in [x-\delta/2, x+\delta/2]} f(y) \geq \frac{f(x)}{2}$. Let $Q = \{a, x-\delta/2, x+\delta/2, b\}$. Since $f \geq 0$,

$$\int_a^b f = \sup_P L(P,f) \geq L(Q,f) \geq \delta \cdot \frac{f(x)}{2} > 0,$$

contradiction.

Remark. *Note how the continuity hypothesis is used. If f is not assumed to be continuous, we could take f to be 1 at a single point and 0 elsewhere as in Exercise 6.1.*

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6.5

Problem. *Suppose f is bounded real function on $[a,b]$, and $f^2 \in \mathcal{R}$ on $[a,b]$. Does it follow that $f \in \mathcal{R}$? Does the answer change if we assume that $f^3 \in \mathcal{R}$?*

Proof. Consider the function $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [a,b] \\ -1 & x \notin \mathbb{Q} \cap [a,b] \end{cases}$. $f^2 \equiv 1$ on $[a,b]$ so $f^2 \in \mathcal{R}$ and $\int f^2 = b-a$. However, $f \notin \mathcal{R}$ as $U(P,f) = b-a$ and $L(P,f) = -1$ for all partition P. Now assume $f^3 \in \mathcal{R}$. Then $f = \phi(f^3)$, where $\phi : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto \sqrt[3]{x}$. Since ϕ is continuous, $f \in \mathcal{R}$ by Theorem 6.11.

Remark. *Note that $f^2 \in \mathcal{R}$ is not enough to conclude $f \in \mathcal{R}$ because the function $x \mapsto x^2$ is not injective so has no continuous inverse on $\mathbb{R}$.*

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6.8

Problem. *Assume that $f \geq 0$ and f decreases monotonically on $[1,\infty)$. Prove that $\int_1^\infty f(x)dx$ converges iff $\sum_{n=1}^\infty f(n)$ converges.*

Proof. Let $n \in \mathbb{N}$ and $P_n = \{1, 2, \cdots, n\}$. Since f decreases monotonically on $[1,\infty)$,

$$L(P_n, f) = \sum_{k=2}^{n} f(k) \leq \int_1^n f(x)dx \leq U(P_n, f) = \sum_{k=1}^{n-1} f(k).$$

It follows from $f \geq 0$ that

$$\sum_{k=2}^{n} f(k) \leq \int_1^z f(x)dx \leq \sum_{k=1}^{n} f(k)$$

for $n \leq z < n+1$. Hence the conclusion. ■

6.11

Problem. *Let α be a fixed increasing function on $[a,b]$. For $u \in \mathcal{R}(\alpha)$, define $||u||_2 = \left\{\int_a^b |u|^2 d\alpha\right\}^{1/2}$. Suppose $f, g, h \in \mathcal{R}(\alpha)$, and prove the triangle inequality*

$$||f-h||_2 \leq ||f-g||_2 + ||g-h||_2.$$

Proof. It suffices to prove $||f+g||_2 \leq ||f||_2 + ||g||_2$ for $f, g \in \mathcal{R}(\alpha)$. Define an inner product on the vector space $\mathcal{R}(\alpha)$ by the map $\mathcal{R}(\alpha) \times \mathcal{R}(\alpha) \to \mathbb{R}$ by $\langle f, g\rangle = \int_a^b fg\,d\alpha$. Verify that this map is an inner product, i.e. the following holds for all $a, b \in \mathbb{R}$, $f, g, h \in \mathcal{R}(\alpha)$

- $\langle af + bg, h\rangle = a\langle f, h\rangle + b\langle g, h\rangle$
- $\langle g, f\rangle = \langle f, g\rangle$
- $\langle f, f\rangle \geq 0$ for all $f \in \mathcal{R}(\alpha)$ and $\langle f, f\rangle = 0$ iff $f = 0$.

Observe that $||f||_2 = \sqrt{\langle f, f\rangle}$. The proposition below shows any vector space with an inner product with the above axioms satisfies the Cauchy-Schwarz inequality and the triangle inequality is a consequence of Cauchy-Schwarz:

$$\begin{aligned} ||f+g||_2^2 &= \langle f+g, f+g\rangle \\ &= ||f||_2^2 + 2\langle f, g\rangle + ||g||_2^2 \\ &\leq ||f||_2^2 + 2||f||_2||g||_2 + ||g||_2^2 \\ &= (||f||_2 + ||g||_2)^2. \end{aligned}$$

■

Proposition. $|\langle x,y\rangle| \le ||x||||y||$ *with equality iff x and y are linearly dependent.*

Proof. Assume $\langle x,y\rangle \neq 0$. Let $\alpha = \text{sign}\langle x,y\rangle \in \{\pm 1\}$ and $z = \alpha y$, so that $\langle x,z\rangle = \langle z,x\rangle = |\langle x,y\rangle|$. For $t \in \mathbb{R}$ we have

$$\begin{aligned} 0 \le \langle x - tz, x - tz\rangle &= ||x||^2 + t^2||y||^2 - 2t|\langle x,y\rangle| \\ &= \frac{1}{||y||^2}\left(t^2 - \frac{2|\langle x,y\rangle|}{||y||^2}t\right) + ||x||^2 \\ &= ||y||^2\left(t - \frac{|\langle x,y\rangle|}{||y||^2}\right)^2 - \frac{|\langle x,y\rangle|^2}{||y||^2} + ||x||^2 \end{aligned}$$

The expression on the right is a quadratic function of t whose absolute minimum occurs at $t = ||y||^{-2}|\langle x,y\rangle|$. Setting t equal to the absolute minimum, we obtain

$$0 \le ||x - tz||^2 = ||x||^2 - ||y||^{-2}|\langle x,y\rangle|^2$$

with equality iff $x = tz = \alpha t y$. ■

Remark. *I lied.* $\mathcal{R}(\alpha)$ *with the inner product defined above is ALMOST an inner product space. Which axiom does not hold? (I told you to verify.) Even though* $\mathcal{R}(\alpha)$ *is not an inner product space, the part of axioms that it fails to satisfy does not affect the proof of the Cauchy-Schwarz inequality. However, the statement that equality iff linearly dependent does not hold for* $\mathcal{R}(\alpha)$.

6.12

Problem. *Suppose* $f \in \mathcal{R}(\alpha)$ *and* $\epsilon > 0$. *Prove that there exists a continuous function* g *on* $[a,b]$ *such that* $||f - g||_2 < \epsilon$.

Proof. Let $\epsilon > 0$ and $M = \sup|f(x)|$. Since $f \in \mathcal{R}(\alpha)$, there exists $P = \{a = x_0, \cdots, x_n = b\}$ such that $U(P,f,\alpha) - L(P,f,\alpha) = \sum_i (M_i - m_i)\Delta\alpha_i < \frac{\epsilon^2}{2M}$. Let g be the piecewise linear function $g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$ for $t \in [x_{i-1}, x_i]$, i.e. g is the line segment connecting $f(x_{i-1})$ and $f(x_i)$ for $1 \le i \le n$. So g is continuous. It is easy to see that $\sup_{x\in[x_{i-1},x_i]} |f(x) - g(x)| \le M_i - m_i$ and the infinmum is 0. So, $L(P, |f - g|^2, \alpha) = 0$ for all P and

$$\begin{aligned} U(P,|f-g|^2,\alpha) &= \sum_i \sup_{x\in[x_{i-1},x_i]} |f(x) - g(x)|^2 \Delta\alpha_i \\ &\le \sum_i (M_i - m_i)^2 \Delta\alpha_i \\ &\le 2M\sum_i (M_i - m_i)\Delta\alpha_i \\ &< \epsilon^2. \end{aligned}$$

Hence $||f - g||_2^2 \le U(P,|f-g|^2,\alpha) < \epsilon^2$. Therefore, $||f - g||_2 < \epsilon$. ■

1

Problem. *Consider $f : [a,b] \to \mathbb{R}$ defined by $f(x) = x^2$. Compute $\int_a^b f(x)dx$ using only the definition of the integral.*

Proof. Assume $b > a \geq 0$. First we need to show $f \in \mathcal{R}$ on $[a,b]$. Consider the partition $P_n = \{a, a+\frac{b-a}{n}, \cdots, b\}$. By computation,

$$U(P_n, f) = \sum_{k=1}^{n} \frac{b-a}{n} \left(a + k \cdot \frac{b-a}{n}\right)^2 = a^2(b-a) + (b-a)^3 \frac{(n+1)(2n+1)}{6n^2} + a(b-a)^2 \frac{n+1}{n}.$$

Similarly, $L(P_n, f) = \sum_{k=0}^{n-1} \frac{b-a}{n} \left(a + k \cdot \frac{b-a}{n}\right)^2 = a^2(b-a) + (b-a)^3 \frac{(n-1)(2n-1)}{6n^2} + a(b-a)^2 \frac{n-1}{n}$. So $U(P_n, f) - L(P_n, f) = \frac{(b-a)(b^2-a^2)}{n} \to 0$ as $n \to \infty$, implying $f \in \mathcal{R}$. Then $\int_a^b f(x)dx = \inf_n U(P_n, f) = \sup_n L(P_n, f) = \frac{b^3-a^3}{3}$. Note that we assume $b > a \geq 0$ so that f is monotonically increasing, $M_i = f(x_i)$, and $m_i = f(x_{i-1})$. Convince yourself this assumption is valid by the continuity of f as the mesh $||P_n|| = \frac{1}{n} \to 0$. ■

2

Problem. *Assume $f : [a,b] \to \mathbb{R}$ satisfies $f \in \mathcal{R}(\alpha)$. Prove that $f^2 \in \mathcal{R}(\alpha)$.*

Proof. Let $\epsilon > 0$ and P be a partition such that $\sum_i (M_i - m_i)\Delta\alpha_i < \frac{\epsilon}{2M}$, where $M = \sup_{x \in [a,b]} |f(x)|$. Observe that $\sup_{x \in [x_{i-1}, x_i]} |f(x)^2| = \sup_{x \in [x_{i-1}, x_i]} |f(x)|^2 = \left(\sup_{x \in [x_{i-1}, x_i]} |f(x)|\right)^2 = M_i^2$ and similarly $\inf_{x \in [x_{i-1}, x_i]} |f(x)^2| = m_i^2$. So

$$U(P, f^2) - L(P, f^2) = \sum_i (M_i^2 - m_i^2)\Delta\alpha_i \leq 2M \sum_i (M_i - m_i)\Delta\alpha_i < \epsilon,$$

implying $f^2 \in \mathcal{R}(\alpha)$. ■